
A PROOF OF THE MARMI-MOUSSA-YOCCOZ CONJECTURE FOR ROTATION NUMBERS OF HIGH TYPE

by

Davoud Cheraghi & Arnaud Chéritat

Abstract. — Marmi Moussa and Yoccoz conjectured that some error function Υ , related to the approximation of the size of Siegel disk by some arithmetic function of the rotation number θ , is a Hölder continuous function of θ with exponent $1/2$. Using the renormalization invariant class of Inou and Shishikura, we prove this conjecture for the restriction of Υ to a class of high type numbers.

1. Introduction

A *Siegel disk* of a complex one dimensional dynamical system $z \mapsto f(z)$ is a maximal open set Δ on which f is conjugate to a rotation on a disk. There is a unique fixed point inside Δ , and its eigenvalue for f is equal to $e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R}$ called the *rotation number*. For a Siegel disk contained in \mathbb{C} and whose fixed point is denoted a , we define its *conformal radius* as the unique $r \in (0, +\infty]$ such that there exists a conformal diffeomorphism $\phi : B(0, r) \rightarrow \mathbb{C}$ with $\phi(0) = a$ and $\phi'(0) = 1$. Since the only self conformal diffeomorphisms of \mathbb{C} and of the unit disk are well known, it is not hard to see that such a ϕ necessarily conjugates f to the rotation $z \mapsto e^{2\pi i\alpha}z$, i.e. it is a *linearizing map*.

Given a holomorphic map f with a fixed point $a \in \mathbb{C}$, if $f'(a) = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R}$ we may wonder if f has a Siegel disk centered on a , i.e. if it is linearizable. This is a subtle question. We will skip its long and interesting history (see [Mil06], Section 11) and jump to the matter needed here.

Given an irrational $\alpha \in \mathbb{R}$, Yoccoz defined the quantity

$$Y(\alpha) = \sum_{n=0}^{+\infty} \beta_{n-1} \log \frac{1}{\alpha_n} \in (0, +\infty].$$

Let us explain what are these numbers. The sequence α_n is the sequence associated to the continued fraction algorithm: $\alpha_0 = \text{Frac}(\alpha)$ and $\alpha_n = \text{Frac}(1/\alpha_n)$. And

2000 Mathematics Subject Classification. — Primary 37F25, Secondary 58D25.

$\beta_n = \alpha_0 \cdots \alpha_n$, with the convention that $\beta_{-1} = 1$. The set of *Brjuno numbers* is the set

$$\mathcal{B} = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid Y(\alpha) < +\infty\}.$$

Consider now the degree 2 polynomial $P_\alpha(z) = e^{2\pi i \alpha} z + z^2$. It is considered as one of the simplest non linear examples one can think of. Depending on α , it may or may not have a Siegel disk. If it does, let $r(\alpha)$ denote the conformal radius. Otherwise, let $r(\alpha) = 0$. Fatou and Julia knew that for α rational, $r(\alpha) = 0$. Yoccoz has completely characterized the set of irrational numbers for which $r(\alpha) > 0$: it coincides with the set of Brjuno numbers. Moreover, he provided a good approximation of $r(\alpha)$: he showed that $r(\alpha) > e^{-Y(\alpha)-C}$, for some constant $C > 0$. He almost showed that $r(\alpha) < e^{-Y(\alpha)+C}$, for some constant $C > 0$. Two proofs for the latter were given by Buff and Chéritat in [BC04] and [BC11].

Yoccoz and Marmi considered then the error function

$$\Upsilon(\alpha) = \log(r(\alpha)) + Y(\alpha).$$

Computer experiments made by Marmi [Mar89] revealed a continuous graph for Υ . It was proved in [BC06] that Υ is the restriction to \mathcal{B} of a continuous function over \mathbb{R} . Then, in [MMY97], Marmi Moussa and Yoccoz conjectured that Υ is in fact $1/2$ -Hölder continuous. Somehow, it cannot be better: it was proved by Buff and Chéritat in [Ché08] that Υ cannot be a -Hölder for $a > 1/2$. Here we will prove that the restriction of Υ to some Cantor set of rotation numbers is indeed $1/2$ -Hölder continuous.

In 2002, Inou and Shishikura constructed a class of maps \mathcal{IS} that is invariant under some renormalization operator [IS06]. Among others, it allows a fine control on the post critical set. It has many applications studied in [Che10, Che12, AC12]. It was also used in 2006 to prove the existence of Julia sets with positive Lebesgue measure [BC12]. Most of those consequences apply to maps that are infinitely renormalizable in the sense of Inou and Shishikura, and the result here will be no exception: we need the rotation number to be of *high type*, i.e. that all its continued fraction entries are at least N , where N is some constant introduced by Inou and Shishikura.

To be more precise, they use *modified continued fractions*, which is almost the same as the standard ones (see the discussion in Section 2). Let $\alpha = a_0 \pm 1/(a_1 \pm 1/(a_2 \pm \cdots))$ be the modified continued fraction, with $a_n \geq 2$, for $\forall n \geq 1$. For $N \geq 2$ let

$$\text{HT}_N = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall n > 0, a_n \geq N\}$$

be the set of high type numbers associated to N . Then there exists $N_0 \geq 2$ such that maps in \mathcal{IS} and with rotation number in HT_{N_0} are infinitely renormalizable.

A value for N_0 is not explicitly computed in [IS06], only its existence is proved.⁽¹⁾ It is also widely believed that there should be an analogous notion of renormalization and a corresponding invariant class, for which there is no assumption to make on the rotation number. However, this has not been found yet.

The object of the present article is to prove the following theorem:

⁽¹⁾Its optimal value is not known but Inou and Shishikura told us that it should not be much less than 20.

Theorem 1. — *With $N = \max\{N_0, 3\}$, the restriction of Υ to HT_N is $1/2$ -Hölder continuous.*

Indeed, we will prove a stronger statement, see Theorem 5.1 in Section 5.1.

The proof is divided into two parts. We first prove a form of Lipschitz dependence of the Inou-Shishikura renormalization of f with respect to $f \in \mathcal{IS}$. More precisely, it is Lipschitz in the direction of the nonlinearity of f , but it is Lipschitz with respect to a modified distance defined by $ds = |d\alpha| \times |\log |\alpha||$ in the direction of rotation. This involves techniques developed by the first author in [Che10, Che12] and utilizes some classical results in the theory of quasi-conformal mappings [Ahl63, Leh76]. We then deduce Hölder continuity of an extension of Υ to \mathcal{IS} , using boundedness of Υ and applying the renormalization operator infinitely many times. This argument is very much in the spirit of [MMY97], and involves arithmetic estimates on the continued fraction that we reproduce here.

Because modified continued fractions are better suited to the Inou-Shishikura renormalization, we will replace the function Y by a variant defined using the modified continued fraction instead of the classical ones, see Section 2. The difference of these two Y functions being $1/2$ -Hölder continuous, the statement of the main theorem is equivalent with either one.

The article is divided as follows. Section 1 is the present introduction. In Section 2, we recall modified continued fractions, Yoccoz's versions of the Brjuno sum, and make comments on different definitions of high type numbers. In Section 3, we recall the Inou and Shishikura class of maps \mathcal{IS} , and the renormalization theorem. In Section 4, we study the dependence of the renormalization on the data. In Section 5, we use renormalization to prove Hölder continuity of Υ .

2. Modified continued fractions

The terminology of *modified continued fractions* might not be standard but it is a very old notion. The connection with our work starts with the discussion in [MMY97]. In their notations, what we call α they call x and they use α as a parameter for a family of continued fraction algorithms. The one we present below corresponds to their continued fraction with parameter $\alpha = 1/2$.

2.1. Definition. — Let $\alpha \in \mathbb{R}$. Let

$$\alpha_0 = d(\alpha, \mathbb{Z}),$$

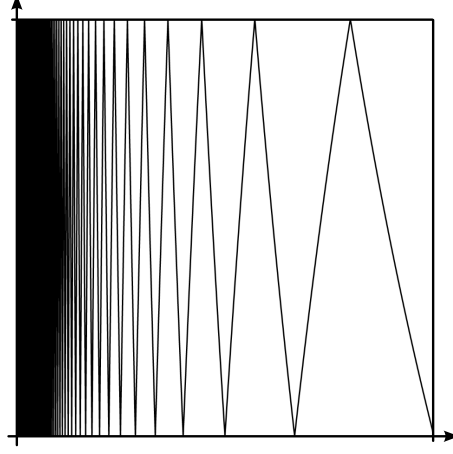
where d denotes the euclidean distance on \mathbb{R} . Let

$$\alpha_{n+1} = d(1/\alpha_n, \mathbb{Z}) = G(\alpha_n)$$

where $G : (0, 1/2] \rightarrow [0, 1/2]$, $x \mapsto d(1/x, \mathbb{Z})$, see Figure 1.

The sequence α_n is defined for all $n \in \mathbb{N}$ iff $\alpha \notin \mathbb{Q}$. Let $a_0 \in \mathbb{Z}$ be such that $\alpha \in [a_0 - 1/2, a_0 + 1/2)$. Let

- s_0 be undefined if $\alpha - a_n = 0$ or $-1/2$.
- $s_0 = -1$ if $\alpha - a_0 \in (-1/2, 0)$,

FIGURE 1. The graph of $G : (0, 1/2] \rightarrow [0, 1/2]$.

- $s_0 = 1$ if $\alpha - a_0 \in (0, 1/2)$.

Similarly, let $a_n \in \mathbb{Z}$ be such that $\alpha_{n-1}^{-1} - a_n \in [-1/2, 1/2)$, and note that $a_n \geq 2$. Define⁽²⁾

- s_n undefined if $\frac{1}{\alpha_{n-1}} - a_n = 0$ or $-1/2$.
- $s_n = -1$ if $\frac{1}{\alpha_{n-1}} - a_n \in (-1/2, 0)$,
- $s_n = 1$ if $\frac{1}{\alpha_{n-1}} - a_n \in (0, 1/2)$.

The map $H_n : \alpha \mapsto \alpha_n$ can be decomposed as

$$H_n = \text{saw} \circ \text{inv} \circ \dots \circ \text{inv} \circ \text{saw},$$

where inv appears n times and saw $n+1$ times, $\text{saw}(x) = d(x, \mathbb{Z})$, whose graph is like a saw, and $\text{inv}(x) = 1/x$. The biggest open intervals on which H_n is a bijection are called *fundamental intervals* (of generation n). They consist in those α with a fixed sequence $(a_0, s_0), \dots, (a_n, s_n)$, called the *symbol* of the interval. See Figure 2. The map H_n is a bijection from each fundamental interval to $(0, 1/2)$. For α irrational, we will denote the n -th generation fundamental interval containing α by

$$I_n(\alpha).$$

Let us recall the following classical result:

⁽²⁾We could have taken a_n to be the floor of $1/\alpha_n$ instead of the nearest integer. There is a simple way to pass from one convention to the other, since (a_n, s_n) in one convention depends solely on (a_n, s_n) in the other. The important thing is to have a way to label the intervals on which the maps H_n defined below are bijections.

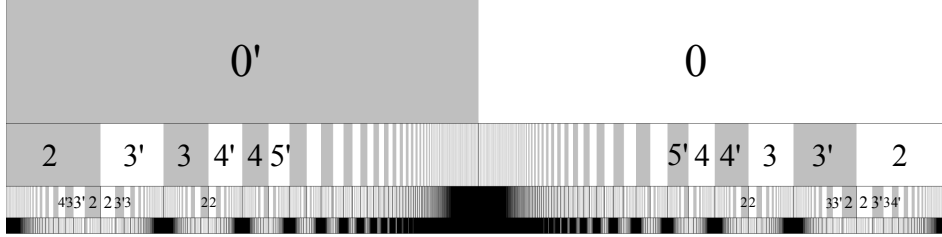


FIGURE 2. Symbolic decomposition of $[-1/2, 1/2]$ according to the modified continued fraction algorithm. An element $x \in [-1/2, 1/2]$ is represented by a vertical line of abscissa x . The rectangle spans the interval $[-1/2, 1/2]$. The top row gives (a_0, s_0) , depending on x . The row below it gives (a_1, s_1) , the next row (a_2, s_2) and the last row (a_3, s_3) . The notation $3'$ is a shorthand for $(3, -)$ whereas 3 means $(3, +)$. The decomposition in fundamental intervals is materialized by the alternating white and gray intervals.

Lemma 2.1. — *There exists $C > 0$ such that for all $n \geq 0$, for all fundamental interval I in the n -th generation, H_n has distortion $\leq C$ on I , that is*

$$\forall x, y \in I, \quad e^{-C} \leq \left| \frac{H'_n(x)}{H'_n(y)} \right| \leq e^C.$$

The union of the fundamental intervals of a given generation is the complement of a countable closed set. Let

$$\beta_{-1} = 1 \quad \text{and} \quad \beta_n = \alpha_0 \cdots \alpha_n.$$

The map H_n is differentiable on each fundamental interval, and

$$H'_n(\alpha) = \pm \frac{1}{\beta_{n-1}^2}.$$

A corollary of bounded distortion⁽³⁾ is the following estimate on the length of the n -th generation fundamental interval containing α :

$$e^{-C} \leq \frac{|I_n(\alpha)|}{\beta_{n-1}^2/2} \leq e^C.$$

2.2. The modified Yoccoz's Brjuno sum. — Originally, Brjuno introduced the sum $B(\alpha) = \sum \frac{\log q_{n+1}}{q_n}$, where p_n/q_n is the sequence of convergents associated to the continued fraction expansion of α . He proved the following. Consider a holomorphic germ defined in a neighborhood of the origin in \mathbb{C} , with expansion $f(z) = e^{2\pi i \alpha} z + \dots$. If α satisfies $B(\alpha) < +\infty$ then f is locally linearizable at 0 [Brj71]. This generalized an earlier result by Siegel [Sie42] under the Diophantine condition. Brjuno's condition

⁽³⁾It can also be proved using an inductive computation on H_n as a Möbius map on fundamental intervals

turned out to be sharp in the quadratic family as proved by Yoccoz [Yoc95]. Yoccoz then defined two functions $Y(\alpha)$, for α irrational, as

$$Y(\alpha) = \sum_{n \geq 0} \beta_{n-1} \log \frac{1}{\alpha_n} \in (0, +\infty]$$

where α_n and β_n are the sequences associated to α in either the classical continued fraction algorithm, in which case we will denote this sum $Y_1(\alpha)$, or the modified continued fraction algorithm, in which case we denote it $Y_{1/2}(\alpha)$. Yoccoz proved that both Y_1 and $Y_{1/2}$ take finite values exactly at the same irrationals, a.k.a. the Brjuno numbers. Indeed, he showed that the difference $Y_1 - Y_{1/2}$ is uniformly bounded from above on irrational numbers. The map Y_1 satisfies the following functional equation:

$$Y_1(\alpha + 1) = Y_1(\alpha) \quad \text{and} \quad \forall \alpha \in (0, 1), \quad Y_1(\alpha) = \log(1/\alpha) + \alpha Y_1(1/\alpha).$$

It is being understood that in both equations, the right hand side is finite if and only if the left hand side is. The map $Y_{1/2}$ satisfies

$$(1) \quad \begin{aligned} Y_{1/2}(\alpha + 1) &= Y_{1/2}(\alpha), \quad Y_{1/2}(-\alpha) = Y_{1/2}(\alpha) \\ \forall \alpha \in (0, 1/2), \quad Y_{1/2}(\alpha) &= \log(1/\alpha) + \alpha Y_{1/2}(1/\alpha). \end{aligned}$$

In [MMY97] (Theorem 4.6), it was proven that $Y_1 - Y_{1/2}$ is Hölder-continuous with exponent $1/2$. It follows that the main theorem of the present article is independent of the choice of the continued fraction expansion, $Y = Y_1$ or $Y = Y_{1/2}$. In the sequel, we use

$$Y = Y_{1/2}.$$

2.3. High type numbers. — For $N \geq 1$, let

$$\text{HT}_N^c = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall n > 0, a_n^c \geq N\}$$

where a_n^c is the sequence in the classical continued fraction expansion of α . For $N \geq 2$ let

$$\text{HT}_N^m = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall n > 0, a_n^m \geq N\}$$

where a_n^m is the sequence in the modified continued fraction. Note that

$$\text{HT}_1^c = \mathbb{R} \setminus \mathbb{Q} = \text{HT}_2^m.$$

There is a simple algorithm to deduce the two sequences (a_n^c) and $((a_n^m, s_n))$ from each other, see [Yoc95]. For $N \geq 2$,

$$\text{HT}_N^c \subset \text{HT}_N^m.$$

Indeed, if all entries $a_n^c \geq 2$ for $n \geq 1$, then the modified expansion has symbols $(a_n^m, s_n) = (a_n^c, +)$.

Note that a similar but opposite inclusion will not hold, because the presence of some $s_n = -$ will imply the presence of a 1 in the sequence a_n^c . So the statement made in our main theorem is slightly more general in its form, which uses $\text{HT}_N = \text{HT}_N^m$, rather than $\text{HT}_N = \text{HT}_N^c$.

3. Renormalization

3.1. Inou-Shishikura class. — Consider the ellipse and the map

$$E := \left\{ x + iy \in \mathbb{C} \mid \left(\frac{x + 0.18}{1.24} \right)^2 + \left(\frac{y}{1.04} \right)^2 \leq 1 \right\}, \text{ and } g(z) := -\frac{4z}{(1+z)^2}.$$

The polynomial $P(z) := z(1+z)^2$ restricted to the domain

$$(2) \quad V := g(\hat{\mathbb{C}} \setminus E),$$

has a fixed point at $0 \in V$ with multiplier 1, a critical point at $-1/3 \in V$ that is mapped to $-4/27$, and another critical point at $-1 \in \mathbb{C} \setminus V$ with $P(-1) = 0$.

Following [IS06], we define the class of maps

$$\mathcal{IS} := \left\{ f := P \circ \varphi_f^{-1}: V_f \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi_f: V \rightarrow V_f \text{ is univalent, } \varphi_f(0) = 0, \varphi'_f(0) = 1 \\ \text{and } \varphi \text{ has a quasi-conformal extension to } \mathbb{C}. \end{array} \right\},$$

and for $A \subseteq \mathbb{R}$,

$$\mathcal{IS}_A := \{ z \mapsto f(e^{2\pi\alpha i} z) : e^{-2\pi\alpha i} \cdot V_f \rightarrow \mathbb{C} \mid f \in \mathcal{IS}_0, \alpha \in A \}.$$

Abusing the notation, $\mathcal{IS}_\beta = \mathcal{IS}_{\{\beta\}}$, for $\beta \in \mathbb{R}$, so $\mathcal{IS}_0 = \mathcal{IS}$. We have the natural projection

$$\pi: \mathcal{IS}_{\mathbb{R}} \rightarrow \mathcal{IS}, \pi(f) = f_0, \text{ where } f_0(z) = f(e^{-2\pi i \alpha} z).$$

The Teichmüller distance between any two elements $f = P \circ \varphi^{-1}$ and $g = P \circ \psi^{-1}$ in \mathcal{IS} is defined as

$$d_{\text{Teich}}(f, g) := \inf \left\{ \log \text{Dil}(\hat{\varphi}_g \circ \hat{\varphi}_f^{-1}) \mid \begin{array}{l} \hat{\varphi}_f \text{ and } \hat{\varphi}_g \text{ are quasi-conformal extensions} \\ \text{of } \varphi_f \text{ and } \varphi_g \text{ onto } \mathbb{C}, \text{ respectively} \end{array} \right\}.$$

This metric is inherited from the one to one correspondence between \mathcal{IS} and the Teichmüller space of $\mathbb{C} \setminus \bar{V}$. It is known that this Teichmüller space with the above metric is a complete metric space. The convergence in this metric implies the uniform convergence on compact sets.

Every map in $\mathcal{IS}_{\mathbb{R}}$ has a neutral fixed point at 0 and a unique critical point at $e^{-2\pi\alpha i} \cdot \varphi(-1/3)$ in $e^{-2\pi\alpha i} \cdot V_f$, where α is the rotation of f at zero. The class $\mathcal{IS}_{\mathbb{R}}$ naturally embeds into the space of univalent maps on the unit disk with a neutral fixed point at 0. Therefore, it is a precompact class in the compact-open topology. In particular, by Koebe distortion theorem, $\{|f''(0)|; f \in \mathcal{IS}_{\mathbb{R}}\}$ is compactly contained in $\mathbb{C} \setminus \{0\}$.

For $h \in \mathcal{IS}_{\mathbb{R}}$ with $h'(0) = e^{2\pi\beta i}$, define $\alpha(h) := \beta$. Also, cp_h denotes the unique critical point of h . When α is small, any map $h(z) = f(e^{2\pi\alpha i} z) \in \mathcal{IS}_{\mathbb{R}}$ with $\alpha = \alpha(h) \neq 0$ has a non-zero fixed point σ_h near 0 in V_h . The σ_h fixed point depends continuously on h and has asymptotic expansion $\sigma_h = -4\pi\alpha i / f''(0) + o(\alpha)$, when h converges to $f \in \mathcal{IS}$ in a fixed neighborhood of 0.

To deal with maps in $\mathcal{IS}_{\mathbb{R}}$ and the quadratic family at the same time, we normalize the quadratic maps to have their critical value at $-4/27$;

$$Q_\alpha(z) := e^{2\pi\alpha i} z + \frac{27}{16} e^{4\pi\alpha i} z^2.$$

According to Inou and Shishikura[IS06], there exists an $\alpha_* > 0$ such that for every $h: V_h \rightarrow \mathbb{C}$ in $\mathcal{IS}_{\mathbb{R}}$ or $h = Q_\alpha: \mathbb{C} \rightarrow \mathbb{C}$, with $\alpha(h) \in (0, \alpha_*]$, there exist a domain $\mathcal{P}_h \subset V_h$ and a univalent map $\Phi_h: \mathcal{P}_h \rightarrow \mathbb{C}$ satisfying the following properties:

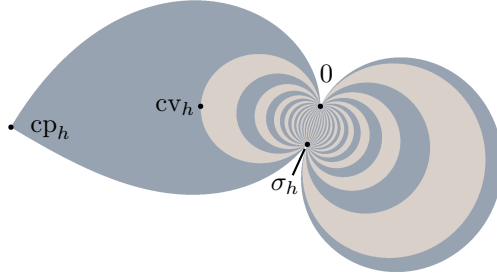


FIGURE 3. The perturbed petal \mathcal{P}_h and the various special points associated to some $h \in \mathcal{IS}_{\mathbb{R}}$. It has been colored so that the map Φ_h sends each strip to an infinite vertical strip of width 1.

- The domain \mathcal{P}_h is bounded by piecewise smooth curves and is compactly contained in V_h . Moreover, it contains cp_h , 0, and σ_h on its boundary.
- $\text{Im } \Phi_h(z) \rightarrow +\infty$ when $z \in \mathcal{P}_h \rightarrow 0$, and $\text{Im } \Phi_h(z) \rightarrow -\infty$ when $z \in \mathcal{P}_h \rightarrow \sigma_h$.
- Φ_h satisfies the Abel functional equation on \mathcal{P}_h , that is,

$$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$

- Φ_h is uniquely determined by the normalization $\Phi_h(cp_h) = 0$. Moreover, the normalized Φ_h depends continuously on h .

Furthermore, it follows from the precompactness of the class $\mathcal{IS}_{\mathbb{R}}$ that there are integers \mathbf{k} and $\hat{\mathbf{k}}$ independents of h such that Φ_h satisfies the following properties⁽⁴⁾.

- There exists a continuous branch of argument defined on \mathcal{P}_h such that

$$\max_{w, w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi\hat{\mathbf{k}}.$$

- $\Phi_h(\mathcal{P}_h)$ contains $\{w \in \mathbb{C} \mid 0 < \text{Re}(w) < \lfloor 1/\alpha \rfloor - \mathbf{k}\}$.

The map $\Phi_h: \mathcal{P}_h \rightarrow \mathbb{C}$ is called the *perturbed Fatou coordinate*, or the *Fatou coordinate* for short, of h . See Figure 3.

⁽⁴⁾ The class \mathcal{IS} is denoted by \mathcal{F}_1 in [IS06]. All above statements, except the existence of uniform $\hat{\mathbf{k}}$ and \mathbf{k} , follow from Theorem 2.1, Main Theorems 1, 3, and Corollary 4.2 in [IS06]. Existence of uniform constants also follow from those results but require some extra work. A detailed treatment of these statements are given in [BC12, Proposition 12].

3.2. Renormalization. — Let $h: V_h \rightarrow \mathbb{C}$ either be in \mathcal{IS}_α or be the quadratic polynomial P_α , with α in $(0, \alpha_*]$. Let $\Phi_h: \mathcal{P}_h \rightarrow \mathbb{C}$ denote the normalized Fatou coordinate of h . Define

$$(3) \quad \begin{aligned} \mathcal{C}_h &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, -2 < \operatorname{Im} \Phi_h(z) \leq 2\}, \\ \mathcal{C}_h^\sharp &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, 2 \leq \operatorname{Im} \Phi_h(z)\}. \end{aligned}$$

By definition, $\operatorname{cv}_h \in \operatorname{int}(\mathcal{C}_h)$ and $0 \in \partial(\mathcal{C}_h^\sharp)$.

Assume for a moment that there exists a positive integer k_h , depending on h , with the following properties:

- For every integer k , with $0 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h^\sharp)$ which is compactly contained in $\operatorname{Dom} h$ and contains 0 on its boundary. We denote this component by $(\mathcal{C}_h^\sharp)^{-k}$.
- For every integer k , with $0 \leq k \leq k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h)$ which has non-empty intersection with $(\mathcal{C}_h^\sharp)^{-k}$, and is compactly contained in $\operatorname{Dom} h$. This component is denoted by \mathcal{C}_h^{-k} .
- The sets $\mathcal{C}_h^{-k_h}$ and $(\mathcal{C}_h^\sharp)^{-k_h}$ are contained in

$$\{z \in \mathcal{P}_h \mid 1/2 < \operatorname{Re} \Phi_h(z) < \lfloor 1/\alpha \rfloor - k - 1/2\}.$$

- The maps $h: \mathcal{C}_h^{-k} \rightarrow \mathcal{C}_h^{-k+1}$, for $2 \leq k \leq k_h$, and $h: (\mathcal{C}_h^\sharp)^{-k} \rightarrow (\mathcal{C}_h^\sharp)^{-k+1}$, for $1 \leq k \leq k_h$, are univalent. The map $h: \mathcal{C}_h^{-1} \rightarrow \mathcal{C}_h$ is a degree two branched covering.

Let k_h be the smallest positive integer satisfying the above four properties, and define

$$S_h := \mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h}.$$

By Abel functional equation, the map

$$(4) \quad \Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \rightarrow \mathbb{C}.$$

projects via $w \mapsto z = \frac{-4}{27}e^{2\pi i w}$ to a well-defined map $\mathcal{R}'(h)$ defined on a set containing 0. Moreover, $\mathcal{R}'(h)$ has asymptotic expansion $e^{2\pi \frac{-1}{\alpha} i} z + O(z^2)$ near 0, See Figure 4.

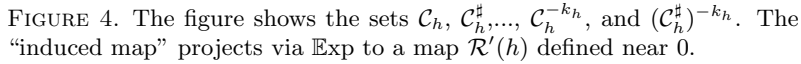
The conjugate map

$$\mathcal{R}(h) := s \circ \mathcal{R}'(h) \circ s^{-1}, \text{ where } s(z) := \bar{z},$$

restricted to the interior of $s(\frac{-4}{27}e^{2\pi i(\Phi_h(S_h))})$, is called the *near-parabolic renormalization* of h by Inou and Shishikura. This map has the form $z \mapsto e^{2\pi \frac{1}{\alpha} i} z + O(z^2)$ near 0. We simply refer to it as the *renormalization* of h . Note that $\mathcal{R}(h)$ is normalized to have the critical value at $-4/27$. For future reference, we define the notation

$$(5) \quad \mathbb{E}xp(\zeta) := \frac{-4}{27}e^{2\pi i \zeta}.$$

The following theorem [IS06, Main theorem 3] states that this definition of renormalization \mathcal{R} can be carried out for certain perturbations of maps in \mathcal{IS} . In particular,



Theorem 3.1 (Inou-Shishikura). — *There exist a constant $\alpha^* > 0$ and a Jordan domain $U \supset \overline{V}$ such that if $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \alpha^*]$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}_{1/\alpha}$. Moreover, with $\mathcal{R}(h) = P \circ \psi^{-1}$, ψ has a univalent extension onto $e^{-2\pi i/\alpha}U$, and the renormalization is a contraction on each slice \mathcal{IS}_α . That is, there exists $\lambda < 1$ such that for all $f, g \in \mathcal{IS}_\alpha$, with $\alpha \in (0, \alpha^*]$, we have*

$$d_{Teich}(\pi \circ \mathcal{R}(f), \pi \circ \mathcal{R}(g)) \leq \lambda d_{Teich}(\pi(f), \pi(g)).$$

4.1. Statement of the main inequalities. — For a map $f \in \mathcal{IS}_{\mathbb{R}}$, or $f = Q_{\alpha}$, that is renormalizable in the sense of Section 3, we define

$$C(f) = \Upsilon(f) - \alpha(f)\Upsilon(\mathcal{R}(f)).$$

(5) The sets \mathcal{C}_h^{-k} and $(\mathcal{C}_h^\sharp)^{-k}$ defined here are (strictly) contained in the sets denoted by V^{-k} and W^{-k} in [BC12]. The set $\Phi_h(\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k})$ is contained in the union

$$D_{-k}^\# \cup D_{-k} \cup D_{-k}'' \cup D_{-k+1}' \cup D_{-k+1} \cup D_{-k+1}^\#$$

in the notations used in [IS06, Section 5.A].

The main purpose of this section is to study the dependence of the above map on the linearity and non-linearity of f , that is, on $\alpha(f)$ and $\pi(f)$. To this end, we define the Riemannian metric $ds = -\log|x||dx|$ on the interval $[-1/2, 1/2]$, where $|dx|$ is the standard Euclidean metric on \mathbb{R} . Let $d_{\log}(x, y)$ denote the induced distance on the interval from this metric. We have $d_{\log}(-1/2, 1/2) = 1 + \log 2 < \infty$.

Let

$$\mathcal{QIS} := \{f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}; \alpha \in (0, \alpha^*)\},$$

where α^* is obtained in Theorem 3.1. This section is devoted to the proofs of the following propositions.

Proposition 4.1. — *There exists a constant K_1 such that*

— *for all $f, g \in \mathcal{IS}_{(0, \alpha^*)}$ we have*

$$|C(f) - C(g)| \leq K_1 [d_{\text{Teich}}(\pi(f), \pi(g)) + d_{\log}(\alpha(f), \alpha(g))];$$

— *for all $\alpha, \beta \in (0, \alpha^*)$ we have*

$$|C(Q_\alpha) - C(Q_\beta)| \leq K_1 d_{\log}(\alpha, \beta).$$

Proposition 4.2. — *There exist constants K_2 and $\lambda < 1$ such that*

— *for all $f, g \in \mathcal{IS}_{(0, \alpha^*)}$ we have*

$$d_{\text{Teich}}(\pi(\mathcal{R}(f)), \pi(\mathcal{R}(g))) \leq \lambda d_{\text{Teich}}(\pi(f), \pi(g)) + K_2 |\alpha(f) - \alpha(g)|;$$

— *for all $\alpha, \beta \in (0, \alpha^*)$ we have*

$$d_{\text{Teich}}(\pi(\mathcal{R}(Q_\alpha)), \pi(\mathcal{R}(Q_\beta))) \leq K_2 |\alpha - \beta|.$$

Proposition 4.3. — *We have*

$$\sup\{|\Upsilon(f)|; f \in \mathcal{QIS}\} < \infty.$$

Proposition 4.4. — *For all $f_0 \in \mathcal{IS} \cup \{Q_0\}$,*

$$\lim_{\alpha \rightarrow 0^+} \Upsilon(z \mapsto f_0(e^{2\pi\alpha i} z)) = \lim_{\alpha \rightarrow 0^-} \Upsilon(z \mapsto f_0(e^{2\pi\alpha i} z)) = \log(4\pi/|f''(0)|).$$

Remark. — Yoccoz [Yoc95] has proved that $\Upsilon(f)$ is uniformly bounded from below, when f is univalent on a fixed domain and $\alpha(f)$ is Brjuno. On the other hand, it is proved in [BC04] that Υ is uniformly bounded from above on the class of quadratic polynomials with Brjuno rotation at 0. Here, we prove that Υ is uniformly bounded from above on the set of maps in $\mathcal{IS}_{(0, \alpha^*)}$ with Brjuno rotation at zero.

In the next subsection we reduce these propositions to some statements on the dependence of the perturbed Fatou coordinate on the map.

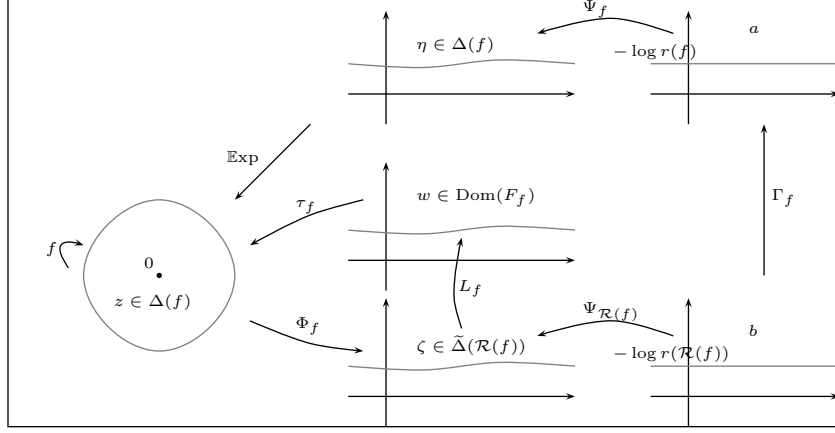


FIGURE 5. A schematic presentation of the maps and coordinates.

4.2. Reduction to the asymptotic expansion. — As Y satisfies Equation (1),

$$(6) \quad C(f) = \log r(f) - \alpha(f) \log r(\mathcal{R}(f)) - \log \alpha(f).$$

For every $f \in QIS$ we let $\Delta(f)$ denote the Siegel disk of f , when f is linearizable. Define

$$\tilde{\Delta}(f) := \mathbb{E} \exp^{-1}(\Delta(f)),$$

where $\mathbb{E} \exp$ is defined in Equation (5), and

$$\mathcal{H}(r) := \{a \in \mathbb{C} \mid \operatorname{Im} a > r\}.$$

By the definition of the conformal radius, there is a conformal map

$$\Psi_f : \mathcal{H}(-\log r(f)) \rightarrow \tilde{\Delta}(f),$$

such that for every $a \in \mathcal{H}(-\log r(f))$ we have

$$\Psi_f(a + 2\pi) = \Psi_f(a) + 1 \text{ and } \Psi_f(a) - \left(\frac{a}{2\pi} - \frac{\mathbf{i}}{2\pi} \log \frac{27}{4}\right) \rightarrow 0, \text{ as } \operatorname{Im} a \rightarrow +\infty.$$

Similarly, we have

$$\Psi_{\mathcal{R}(f)} : \mathcal{H}(-\log r(\mathcal{R}(f))) \rightarrow \tilde{\Delta}(\mathcal{R}(f)),$$

satisfying the above conditions. See Figure 5.

Recall the perturbed Fatou coordinate $\Phi_f : \mathcal{P}_f \rightarrow \mathbb{C}$ of f that conjugates the iterates of f to that of the translation by one. Using the relation $\Phi_f \circ f(z) = \Phi_f(z) + 1$, we can extend the map Φ_f^{-1} to a covering map from $\tilde{\Delta}(\mathcal{R}(f)) \rightarrow \Delta(f) \setminus \{0\}$. This provides us with a conformal isomorphism

$$\Gamma_f : \mathcal{H}(-\log r(\mathcal{R}(f))) \rightarrow \mathcal{H}(-\log r(f)),$$

defined as the lift $\Psi_f^{-1} \circ \mathbb{E} \exp^{-1} \circ \Phi_f^{-1} \circ \Psi_{\mathcal{R}(f)}$. Since $\operatorname{Im} \Gamma_f(b) \rightarrow +\infty$, as $\operatorname{Im} b \rightarrow +\infty$, Γ_f must be a linear map. From (6) we conclude that

$$(7) \quad \Gamma_f(b) = \alpha b - \mathbf{i} \log \alpha - \mathbf{i} C(f).$$

By virtue of this isomorphism, the relation between the two conformal radii can be understood from the asymptotic expansion of Γ_f near infinity. To understand the dependence of $C(f)$ on f , we analyze how the asymptotic expansion of Γ_f depends on f .

Let f be a map as above, and let σ_f denote the non-zero fixed point of f with asymptotic expansion $\sigma_f = -4\pi\alpha(f)\mathbf{i}/f''(0) + o(\alpha(f))$. We consider the covering map

$$(8) \quad \tau_f(w) := \frac{\sigma_f}{1 - e^{-2\pi\alpha(f)\mathbf{i}w}} : \mathbb{C} \rightarrow \hat{\mathbb{C}} \setminus \{0, \sigma_f\}.$$

with deck transformation generated by $w \rightarrow w + 1/\alpha(f)$. We have $\tau_f(w) \rightarrow 0$, as $\text{Im } w \rightarrow +\infty$, and $\tau_f(w) \rightarrow \sigma$, as $\text{Im } w \rightarrow -\infty$.

The map f lifts under τ_f to a map F_f defined on $\text{Dom } F_f := \tau_f^{-1}(\text{Dom } f)$. One can see that $F_f(w) \sim w + 1$ as $\text{Im } w \rightarrow \infty$, and $\tau_f(F_f(w)) = f(\tau_f(w))$, for all $w \in \text{Dom } F_f$.

We decompose the map $\Phi_f^{-1} : \{\xi \in \mathbb{C} \mid \text{Re } \xi \in (0, 1/\alpha(f) - \mathbf{k})\} \rightarrow \mathcal{P}_f$ into two maps as $\Phi_f^{-1} = \tau_f \circ L_f$, where the inverse branch of τ_f is chosen so that $\tau_f^{-1}(\mathcal{P}_f)$ separates 0 and $1/\alpha(f)$. The conformal map L_f satisfies

$$L_f(\xi + 1) = F_f(L_f(\xi)), \tau_f(L_f(1)) = -4/27, \text{ and } L_f'(\xi) \rightarrow 1 \text{ as } \text{Im } \xi \rightarrow +\infty.$$

Define the complex constant

$$\ell_f := \text{Im} \lim_{\text{Im } \xi \rightarrow \infty} (L_f(\xi) - \xi).$$

A byproduct of the proof of the next lemma is that this quantity is finite (see Equation (16)).

Lemma 4.5. — *There exists a constant K_3 such that*

– *for all $f, g \in \mathcal{IS}_{(0, \alpha^*)}$*

$$|\alpha(f)\ell_f - \alpha(g)\ell_g| \leq K_5 [d_{\text{Teich}}(\pi(f), \pi(g)) + d_{\log}(\alpha(f), \alpha(g))];$$

– *for all $\alpha, \beta \in (0, \alpha^*)$*

$$|\alpha\ell_{Q_\alpha} - \beta\ell_{Q_\beta}| \leq K_5 d_{\log}(\alpha, \beta).$$

In a moment we will be able to prove Proposition 4.1 from the above lemma. Before this can happen, we need to relate d_{Teich} on \mathcal{IS} to some pointwise differences. The proof uses K oebe distortion Theorem and Lehto Majorant principle, [Leh87]. Let Σ denote the space of all one to one holomorphic (univalent) functions f definite on the unit disk with $f(0) = 0$ and $f'(0) = 1$. Also, let $\Sigma_k \subseteq \Sigma$, for $k \in [0, 1]$, be the set of all univalent functions that have a quasi-conformal extension onto \mathbb{C} with complex dilatation $\text{dil}(f) \leq k$. A complex valued functional $\Phi : \Sigma \rightarrow \mathbb{C}$ is called analytic, if it is a holomorphic function of finitely many coefficients of f and the values of $f, f', \dots, f^{(m)}$ at finitely many given points of the unit disk. For example, $f \rightarrow f''(z)$, is an analytic functional, for a given z in the unit disk.

Theorem 4.6 (Distortion Estimates). —

Köbe : For every $f \in \Sigma$,

$$\frac{1 - |z|}{1 + |z|^3} \leq |f'(z)| \leq \frac{1 + |z|}{1 - |z|^3}$$

Lehto : If Φ is an analytic functional on Σ that vanishes for the identity mapping, then

$$\max_{f \in \Sigma_k} |\Phi(f)| \leq k \max_{f \in \Sigma} |\Phi(f)|.$$

Recall the domain V defined in (2).

Lemma 4.7. — Let $V' \subset V$ be a simply connected domain with the modulus of $V \setminus V'$ equal to ε . For every $\varepsilon > 0$ there exists a constant K_4 such that for all $f, g \in \mathcal{IS}_{\mathbb{R}}$, all $z \in V_g \cap V_f$ with $\varphi_f^{-1}(z), \varphi_g^{-1}(z) \in V'$ we have

$$|f(z) - g(z)| \leq K_4 |z| [|\alpha(f) - \alpha(g)| + d_{\text{Teich}}(\pi(f), \pi(g))].$$

In particular, for a possibly larger constant K_4 we have

$$|f''(0) - g''(0)| \leq K_4 [|\alpha(f) - \alpha(g)| + d_{\text{Teich}}(\pi(f), \pi(g))].$$

Proof. — First an elementary reduction as follows,

$$\begin{aligned} |f(z) - g(z)| &\leq |\varphi_f^{-1}(z) - \varphi_g^{-1}(z)| \cdot \sup_{w \in V'} |P'(w)| \\ &\leq |\varphi_f^{-1} \circ \varphi_g \circ \varphi_g^{-1}(z) - \varphi_f^{-1} \circ \varphi_f \circ \varphi_g^{-1}(z)| \cdot \sup_{w \in V'} |P'(w)| \\ &\leq \inf_{w \in V'} |\varphi'_f(w)| \cdot |\varphi_g(\varphi_g^{-1}(z)) - \varphi_f(\varphi_g^{-1}(z))| \cdot \sup_{w \in V'} |P'(w)| \\ &\leq A_1 |\varphi_g(\varphi_g^{-1}(z)) - \varphi_f(\varphi_g^{-1}(z))| \end{aligned}$$

where, A_1 is a constant depending only on ε , by Köbe distortion Theorem.

On the other hand with $\varphi := \varphi'_g(0)\varphi_f/\varphi'_f(0)$ and $w = \varphi_g^{-1}(z) \in V'$,

$$|\varphi_f(w) - \varphi_g(w)| \leq |\varphi_f(w) - \varphi(w)| + |\varphi(w) - \varphi_g(w)|$$

For the first term in the right hand side,

$$\begin{aligned} |\varphi_f(w) - \varphi(w)| &\leq |\varphi_f(w)| \cdot \left| 1 - \frac{\varphi'_g(0)}{\varphi'_f(0)} \right| \\ &\leq |\varphi_f \circ \varphi_g^{-1}(z)| \cdot \left| 1 - \frac{\varphi'_g(0)}{\varphi'_f(0)} \right| \\ &\leq A_2 |z| \cdot |\alpha(f) - \alpha(g)| \end{aligned}$$

for some constant A_2 , by Köbe distortion Theorem.

For the second term,

$$\begin{aligned} |\varphi(w) - \varphi_g(w)| &\leq \sup_{w' \in V'} |\varphi'(w')| \cdot |w - \varphi^{-1} \circ \varphi_g(w)| \\ &\leq \sup_{w' \in V'} |\varphi'(w')| \cdot \text{dil}(\hat{\varphi}^{-1} \circ \hat{\varphi}_g) \cdot |w| \\ &= A_3 d_{\text{Teich}}(\pi(f), \pi(g)) |z|, \end{aligned}$$

for some constant A_3 , by K oebe distortion Theorem and Lehto Majorant principle. Also, note that $\text{dil} = \frac{\text{Dil}-1}{\text{Dil}+1} \leq \log \text{Dil}$.

The second estimate in the lemma is obtained from the first one and the Cauchy integral formula. \square

Proof of Proposition 4.1 assuming Lemma 4.5. — The plan is to find $C(f)$ in terms of ℓ_f , by comparing the asymptotic expansion (7) to the one obtained from the changes of coordinates in the renormalization.

We use the following coordinates

$$b \in \mathcal{H}(-\log r(\mathcal{R}(f))), a \in \mathcal{H}(-\log r(f)), \zeta \in \tilde{\Delta}(\mathcal{R}(f)), \eta \in \tilde{\Delta}(f), z \in \Delta(f).$$

$$\begin{aligned} \text{Im } a &= 2\pi \text{Im } \eta + \log \frac{27}{4} \\ &= -\log |z| \\ &= -[\log |\sigma_f| - \log |1 - e^{-2\pi\alpha(f)L_f(\zeta)\mathbf{i}}|] \\ &\approx -\log |\sigma_f| + 2\pi\alpha(f) \text{Im } L_f(\zeta) \\ &= -\log |\sigma_f| + 2\pi\alpha(f) \text{Im } \zeta + 2\pi\alpha(f) \text{Im } \ell_f \\ &= -\log \alpha(f) - \log \frac{|f''(0)|}{4\pi} + \text{Im } b \cdot \alpha(f) - \alpha(f) \log \frac{27}{4} + 2\pi\alpha(f) \text{Im } \ell_f \end{aligned}$$

Now, comparing the above expression to (7), we have

$$(9) \quad C(f) = -\log |f''(0)| + \log(4\pi) - 2\pi\alpha\ell_f + \alpha(f) \log \frac{27}{4}.$$

Now the estimates in Lemmas 4.7 and 4.5 imply Proposition 4.1. Note that by the discussion in Subsection 3.1 $|f''(0)|$ is uniformly away from 0. \square

The proof of Lemma 4.5 can be divided into two parts using the triangle inequality. Given maps f and g , define the new map $h(z) := f(e^{2\pi(\alpha(g)-\alpha(f))\mathbf{i}} \cdot z)$, for z in $e^{2\pi(\alpha(f)-\alpha(g))\mathbf{i}} \cdot \text{Dom } f$. Then

$$(10) \quad |\alpha(f)\ell_f - \alpha(g)\ell_g| \leq |\alpha(f)\ell_f - \alpha(h)\ell_h| + |\alpha(h)\ell_h - \alpha(g)\ell_g|.$$

By virtue of the above inequality, Lemma 4.5 follows from the next two lemmas.

Lemma 4.8. — *There exists a constant K_5 such that for all $g, h \in \mathcal{IS}_{(0, \alpha^*]}$ with $\alpha(h) = \alpha(g)$, we have*

$$|\alpha(h)\ell_h - \alpha(g)\ell_g| \leq K_5 d_{\text{Teich}}(\pi(h), \pi(g)).$$

Note that $\pi(f) = \pi(h)$.

Lemma 4.9. — *There exists a constant K_6 such that*

– *for all $f, h \in \mathcal{IS}_{(0, \alpha^*]}$, where $f = h(e^{2\pi(\alpha(f)-\alpha(h))\mathbf{i}} \cdot z)$, we have*

$$|\alpha(f)\ell_f - \alpha(h)\ell_h| \leq K_6 d_{\log}(\alpha(f), \alpha(h));$$

– *for all $\alpha, \beta \in (0, \alpha^*]$ we have*

$$|\alpha\ell_{Q_\alpha} - \beta\ell_{Q_\beta}| \leq K_6 d_{\log}(\alpha, \beta).$$

Before we embark on proving the above two lemmas, we need some basic estimates on F_f , as well as its dependence on the non-linearity of f and on $\alpha(f)$.

4.3. The lift F_f and its dependence on $\pi(f)$ and $\alpha(f)$. — From the domain of analyticity improvement in Theorem 3.1, one obtains a uniform bound on the number of pre-images needed to take in the definition of the renormalization. This is stated in the next lemma.

Lemma 4.10. — *There exists $K_7 \in \mathbb{Z}$ such that for all $f \in \mathcal{QLS}$, $k_f \leq K_7$.*

Proof. — Comparing with the rotation of angle α , there exists an integer $j \in [\mathbf{k}, \mathbf{k}+3]$ such that for every $w \in \mathcal{C}_f \cup \mathcal{C}_f^\sharp$ close to zero, there exists a unique inverse orbit $w, f^{-1}(w), \dots, f^{-j}(w)$, contained in a neighborhood of zero, with $f^{-j}(w) \in \mathcal{P}_f$. As k_f is the smallest positive integer with $\mathcal{C}_f^{-k_f} \cup (\mathcal{C}_f^\sharp)^{-k_f} \subseteq \mathcal{P}_f$, we have $k_f \geq \mathbf{k}$.

Now let $\gamma : [0, \infty) \rightarrow \Phi_f(\mathcal{P}_f)$ be a continuous curve satisfying the following.

- $\Phi_f \circ f^{\circ k_f} \circ \Phi_f^{-1}(\gamma(t)) = s + ti$, for $t \in (-2, \infty)$, and a fixed $s \in [1/2, 3/2]$;
- $\text{diam} \{\text{Re } \gamma(t) \mid t \in [0, \infty)\} \geq k_f - \mathbf{k} - 4$.

The curve $\hat{\gamma} := \text{Exp} \circ \gamma$ lands at zero, and part of it must spiral at least $k_f - \mathbf{k} - 4$ times around zero. By the definition of renormalization, $\mathcal{R}(f)$ maps $\hat{\gamma}$ to a straight ray landing at zero. On the other hand, by Theorem 3.1, $\mathcal{R}(f)$ is of the form $P \circ \psi(e^{2\pi\alpha i \cdot})$ with $\psi : V \rightarrow \mathbb{C}$ extending univalently over the larger domain U which compactly contains V . By Kőbe distortion Theorem, the total spiralling of the pull back of a straight ray landing at zero under $\mathcal{R}(f)$ must be uniformly bounded by some constant independent of f . That is, $k_f - \mathbf{k} - 4$ is uniformly bounded from above. \square

Lemma 4.11. — *There exists a constant K_8 such that for every $f \in \mathcal{QLS}$, we have*

$$|\tau_f(w)| \leq K_8 \frac{\alpha}{e^{2\pi\alpha \text{Im } w} - 1}, \text{ if } \text{Im } w > 0,$$

$$|\tau_f(w) - \sigma_f| \leq K_8 \frac{\alpha e^{2\pi\alpha \text{Im } w}}{1 - e^{2\pi\alpha \text{Im } w}}, \text{ if } \text{Im } w < 0.$$

Proof. — This is obtained from the pre-compactness of the class \mathcal{IS} . See the proof of Lemma 4.12 for further details. \square

From now on, we fix the inverse branch log so that its imaginary part belongs to $[0, 2\pi]$.

Lemma 4.12. — *There exists a constant K_9 such that for all f in \mathcal{QLS} , the map F_f satisfies the following estimates.*

- 1) *For all $w \in \mathbb{C} \setminus B(\mathbb{Z}/\alpha, K_9)$, we have*

$$|F_f(w) - (w + 1)| \leq 1/4, \text{ and } |F'_f(w) - 1| \leq 1/4.$$

- 2) *For all $w \in \mathbb{C} \setminus B(\mathbb{Z}/\alpha, K_9)$ with $\text{Im } w > 0$*

$$|F_f(w) - (w + 1)| \leq K_9 |\tau_f(w)|, \text{ and } |F'_f(w) - 1| \leq K_9 |\tau_f(w)|.$$

3) For all $w \in \mathbb{C} \setminus B(\mathbb{Z}/\alpha, K_9)$ with $\text{Im } w < 0$

$$|F_f(w) - w + \frac{1}{2\pi\alpha(f)\mathbf{i}} \log f'(\sigma_f)| \leq K_9 |\tau_f(w) - \sigma_f|,$$

$$|F'_f(w) - 1| \leq K_9 |\tau_f(w) - \sigma_f|.$$

Lemma 4.13. — There exists a constant K_{10} such that for all h, g in $\mathcal{IS}_{(0, \alpha^*]}$ with $\alpha(h) = \alpha(g)$, the maps F_h and F_g satisfy the following estimate.

1) When $\text{Im } w > 0$,

$$|F_h(w) - F_g(w)| \leq K_{10} d_{\text{Teich}}(\pi(h), \pi(g)) |\tau_h(w)|.$$

2) When $\text{Im } w < 0$,

$$|F_h(w) - F_g(w)| \leq K_{10} d_{\text{Teich}}(\pi(h), \pi(g)) |\tau_h(w) - \sigma_h|.$$

Lemma 4.14. — There exists a constant K_{11} such that the following holds. Given $f \in \mathcal{IS}_0 \cup \{z + z^2\}$ and $\alpha \in (0, \alpha^*]$, let $f_\alpha(z) := f(e^{2\pi\alpha\mathbf{i}}z)$, $\tau_\alpha := \tau_{f_\alpha}$, $\sigma_\alpha := \sigma_{f_\alpha}$, and $F_\alpha := F_{f_\alpha}$. Then for all $w \in \mathbb{C} \setminus B(\mathbb{Z}/\alpha, K_9)$ we have

1) when $\text{Im } w > 0$,

$$|\frac{d}{d\alpha} F_\alpha(w)| \leq K_{11} |\tau_\alpha(w)|;$$

2) when $\text{Im } w < 0$,

$$|\frac{d}{d\alpha} F_\alpha(w)| \leq K_{11} |\tau_\alpha(w) - \sigma_\alpha|.$$

It is an easy application of the Kőbe distortion Theorem, and the inequality in Lemma 4.12-1 that the derivative of L_f has absolute value uniformly bounded from above and below. As this will be frequently used in the sequel, we present it as a lemma

Lemma 4.15. — There exists a constant K_{12} such that for every $f \in \mathcal{QLS}$, and every $w \in \mathbb{C}$ with $\text{Re}(w) \in (1/2, [1/\alpha(f)] - k - 1/2)$, we have $1/K_{12} \leq |L'_f| \leq K_{12}$.

Proof of Lemmas 4.12, 4.13, and 4.14. — Following [Shi98], the map f can be written as

$$(11) \quad f(z) = z + z(z - \sigma_f)u_f(z),$$

where u_f is a holomorphic function defined on $\text{Dom } f$, and is non-vanishing at zero and σ_f . Differentiating the above equation at zero, we obtain

$$\sigma_f = (1 - e^{2\pi\alpha(f)\mathbf{i}})/u_f(0).$$

Recall the covering map τ_f , and observe that for every $w \in \mathbb{C}$ with $\tau_f(w) \in \text{Dom } f$, F_f is given by the formula

$$F_f(w) = w + \frac{1}{2\pi\alpha(f)\mathbf{i}} \log \left(1 - \frac{\sigma_f u_f(z)}{1 + z u_f(z)} \right), \text{ with } z = \tau_f(w).$$

Clearly, F_f is periodic of period $1/\alpha(f)$. See that

$$\begin{aligned} |F_f(w) - (w + 1)| &= \left| \frac{1}{2\pi\alpha(f)\mathbf{i}} \log \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) - 1 \right| \\ &= \left| \frac{1}{2\pi\alpha(f)\mathbf{i}} \log \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) - \frac{\log e^{2\pi\alpha(f)\mathbf{i}}}{2\pi\alpha(f)\mathbf{i}} \right| \\ &= \frac{1}{2\pi\alpha(f)} \left| \log \left(\left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) e^{-2\pi\alpha(f)\mathbf{i}} \right) \right|. \end{aligned}$$

Moreover, replacing σ_f by the above expression,

$$\begin{aligned} \left| \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) e^{-2\pi\alpha(f)\mathbf{i}} - 1 \right| &= \left| \left(1 - \frac{\sigma_f u_f(z)}{1 + zu_f(z)} \right) - e^{2\pi\alpha(f)\mathbf{i}} \right| \\ &= |1 - e^{2\pi\alpha(f)\mathbf{i}}| \left| 1 - \frac{u_f(z)}{(1 + zu_f(z))u_f(0)} \right| \\ &\leq 2\pi\alpha(f) \left| 1 - \frac{u_f(z)}{(1 + zu_f(z))u_f(0)} \right|. \end{aligned}$$

On the other hand, by the pre-compactness of the class \mathcal{IS} , there exists a constant C (independent of f and α) such that the set $\tau_f(\mathbb{C} \setminus B(\mathbb{Z}/\alpha, C))$ is contained in a given neighborhood of zero. Since the set of maps u_f forms a pre-compact class in the compact-open topology, the above estimates imply the first estimate in Lemma 4.12-1. The second estimate follows from the first one using Cauchy Integral Formula, and replacing C by $C + 1$.

For the finer estimate in Part 2, we further estimate the above expression as

$$\left| 1 - \frac{u_f(z)}{(1 + zu_f(z))u_f(0)} \right| \leq 2 \left| \frac{h'_f(0) - u_f^2(0)}{u_f(0)} \right| |z| \leq C |\tau_f(w)|$$

for z in φ_f of a fixed compact subset of V , by the pre-compactness of \mathcal{IS} . The other estimate also follows from Cauchy Integral Formula.

Next, we estimate F_f near the lower end. Differentiating Equation (11) at σ_f , we obtain

$$\sigma_f u_f(\sigma_f) = f'(\sigma_f) - 1.$$

Moreover, the multipliers of f at these two fixed points are related by the holomorphic fixed-point formula

$$\frac{1}{1 - f'(0)} + \frac{1}{1 - f'(\sigma_f)} = \frac{1}{2\pi\mathbf{i}} \oint_{\gamma} \frac{dz}{f(z) - z},$$

where γ is a small closed loop containing only these fixed points of f . In particular, when $\alpha(f)$ is a small positive number, the rotation at σ_f is a small negative number,

with their ratio tending to -1 . We have,

$$\begin{aligned} |F_f(w) - w + \frac{1}{2\pi\alpha(f)\mathbf{i}} \log f'(\sigma_f)| \\ = |\frac{1}{2\pi\alpha(f)\mathbf{i}} \log(1 - \frac{\sigma_f u_f(z)}{1 + z u_f(z)}) + \frac{1}{2\pi\alpha(f)\mathbf{i}} \log f'(\sigma_f)| \\ = \frac{1}{2\pi\alpha(f)} |\log((1 - \frac{\sigma_f u_f(z)}{1 + z u_f(z)}) f'(\sigma_f))|, \end{aligned}$$

and, using the above expression for σ_f ,

$$\begin{aligned} |(1 - \frac{\sigma_f u_f(z)}{1 + z u_f(z)}) f'(\sigma_f) - 1| \\ \leq |f'(\sigma_f)| \left| \left(\frac{1 + (z - \sigma_f) u_f(z)}{1 + z u_f(z)} \right)' \right|_{z=\sigma_f} \cdot |z - \sigma_f| \\ \leq \left| \frac{u_f(\sigma_f)(f'(\sigma_f) - 1) + \sigma_f u'_f(\sigma_f)}{f'(\sigma_f)} \right| \cdot |z - \sigma_f| \\ \leq \left(|f'(\sigma_f) - 1| \left| \frac{u_f(\sigma_f)}{f'(\sigma_f)} \right| + \left| \frac{1 - e^{2\pi\alpha(f)\mathbf{i}}}{u_f(0)} \right| \left| \frac{u'_f(\sigma_f)}{f'(\sigma_f)} \right| \right) |z - \sigma_f| \\ \leq C \left(\frac{1}{2\pi\mathbf{i}} \log f'(\sigma_f) + \alpha(f) \right) |z - \sigma_f|. \end{aligned}$$

Hence, using the holomorphic index formula,

$$\begin{aligned} |F_f(w) - w + \frac{1}{2\pi\alpha\mathbf{i}} \log(\frac{1}{f'(\sigma_f)})| &\leq \frac{1}{2\pi\alpha(f)} C \left(\frac{1}{2\pi\mathbf{i}} \log f'(\sigma_f) + \alpha(f) \right) |z - \sigma_f| \\ &\leq C \left(\frac{1}{\pi} + \frac{1}{2\pi} \left| \int_{\gamma} \frac{dz}{f(z) - z} \right| \right) |\tau_f(w) - \sigma_f| \\ &\leq C' |\tau_f(w) - \sigma_f|. \end{aligned}$$

To prove Lemma 4.13, assume $\text{Im } w > 0$ and let $z = \tau_f(w)$ and $z' = \tau_g(w)$. We have,

$$\begin{aligned} |F_f(w) - F_g(w)| &\leq \frac{1}{2\pi\alpha(f)} \left| \log \left(1 - \frac{\sigma_f u_f(z)}{1 + z u_f(z)} \right) - \log \left(1 - \frac{\sigma_g u_g(z')}{1 + z' u_g(z')} \right) \right| \\ &\leq C \left| \frac{u_f(z)}{(1 + z u_f(z)) u_f(0)} - \frac{u_g(z')}{(1 + z' u_g(z')) u_g(0)} \right| \\ &\leq C \left(\left| \frac{u_f(z)}{(1 + z u_f(z)) u_f(0)} - \frac{u_g(z)}{(1 + z u_g(z)) u_g(0)} \right| \right. \\ &\quad \left. + \left| \frac{u_g(z)}{(1 + z u_g(z)) u_g(0)} - \frac{u_g(z')}{(1 + z' u_g(z')) u_g(0)} \right| \right) \\ &\leq C' (|z| d_{\text{Teich}}(\pi(f), \pi(g)) + \frac{1}{u_g(0)} |z - z'|) \\ &\leq C'' |\tau_f(w)| d_{\text{Teich}}(\pi(f), \pi(g)). \end{aligned}$$

The estimate for $\text{Im } w < 0$ is proved similarly, using the inequality

$$\left| \int_{\gamma} \frac{dz}{f(z) - z} - \int_{\gamma} \frac{dz}{g(z) - z} \right| \leq C \, d_{\text{Teich}}(\pi(f), \pi(g)),$$

for some constant C independent of α , that can be obtained from Lemma 4.7.

Finally, we prove Lemma 4.14. Again, fix $w \in B(\mathbb{Z}/\alpha, K_9)$ with $\text{Im } w > 0$ and let $z(\alpha) := \tau_{f_\alpha}(w)$. Then,

$$\begin{aligned} \left| \frac{d}{d\alpha} F_\alpha(w) \right| &= \left| \frac{d}{d\alpha} \left(\frac{1}{2\pi\alpha(f)\mathbf{i}} \log \left(1 - \frac{\sigma_\alpha u_{f_\alpha}(z(\alpha))}{1 + z(\alpha)u_{f_\alpha}(z(\alpha))} \right) \right) \right| \\ &\leq C \left| \frac{d}{d\alpha} \left(\frac{u_{f_\alpha}(z(\alpha))}{(1 + z(\alpha)u_{f_\alpha}(z(\alpha)))u_{f_\alpha}(0)} \right) \right| \\ &\leq C' |z(\alpha)| \end{aligned}$$

The other case is proved similarly. \square

4.4. Proof of the estimates on the asymptotic expansion. — Here we shall show that Lemma 4.13 implies Lemma 4.8, and Lemma 4.14 implies Lemma 4.9.

Proof of Lemma 4.8. — Within this proof all the constants A_1, A_2, \dots depend only on the class \mathcal{QLS} .

Given f , let $v'_f := L_f(1)$; a lift of $-4/27$ under τ_f that is a critical value of F_f . By the pre-compactness of the class \mathcal{IS} , there exists a non-negative integer s such that for all $f \in \mathcal{QLS}$ the vertical line through $v''_f := F_f^{\circ s}(v'_f)$ is contained in $\mathbb{C} \setminus B(\mathbb{Z}/\alpha, K_9)$. The distance between any two such values, using appropriate inverse branches of τ_g and τ_h , can be estimated as follows.

$$\begin{aligned} |v''_g - v''_h| &= |\tau_g^{-1}(g^{\circ s}(-4/27)) - \tau_h^{-1}(h^{\circ s}(-4/27))| \\ &\leq |\tau_g^{-1}(g^{\circ s}(-4/27)) - \tau_g^{-1}(h^{\circ s}(-4/27))| \\ &\quad + |\tau_g^{-1}(h^{\circ s}(-4/27)) - \tau_h^{-1}(h^{\circ s}(-4/27))| \\ (12) \quad &\leq \frac{1}{2\pi\alpha} \left(\log \left(1 - \frac{1 - e^{2\pi\alpha\mathbf{i}}}{g^{\circ s}(-4/27)u_g(0)} \right) - \log \left(1 - \frac{1 - e^{2\pi\alpha\mathbf{i}}}{h^{\circ s}(-4/27)u_g(0)} \right) \right) \\ &\quad + \frac{1}{2\pi\alpha} \left(\log \left(1 - \frac{1 - e^{2\pi\alpha\mathbf{i}}}{h^{\circ s}(-4/27)u_g(0)} \right) - \log \left(1 - \frac{1 - e^{2\pi\alpha\mathbf{i}}}{h^{\circ s}(-4/27)u_h(0)} \right) \right) \\ &\leq A_1 \, d_{\text{Teich}}(\pi(h), \pi(g)), \end{aligned}$$

for a constant C independent of g, h , and α .

By virtue of Lemma 4.12-1, the vertical line $v''_f + t\mathbf{i}$, for $t \in \mathbb{R}$, is mapped away from itself under F_f . Hence, we may define a homeomorphism $H_f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{C}$ as below.

$$H_f(s, t) := (1 - s)(v''_f + t\mathbf{i}) + sF_f(v''_f + t\mathbf{i}).$$

The asymptotic behavior of this map is given by

$$(13) \quad H_f(s, t) \approx s + t\mathbf{i} + v''_f, \text{ as } t \rightarrow \infty.$$

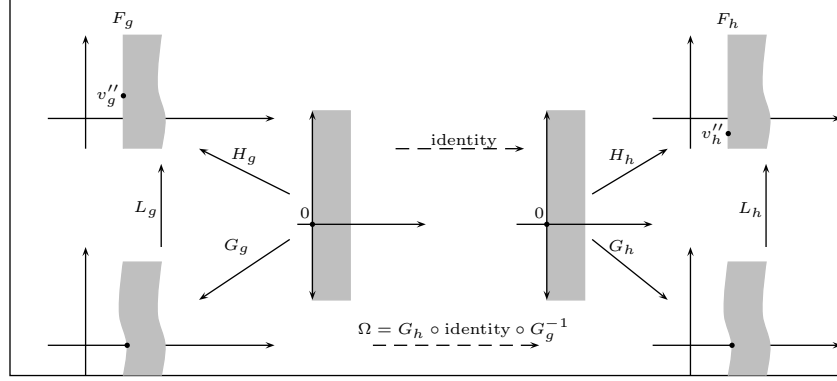


FIGURE 6. The maps H_g and H_h are models for the Fatou coordinates L_g and L_h , respectively. The map L_g is compared to L_h by studying $\Omega = G_h \circ G_g^{-1}$ using its complex dilatation. Besides, the complex dilatation of Ω corresponds to the one for $H_h \circ H_g^{-1}$.

Using the complex notation $\zeta = s + it$, the first partial derivatives of H are given by

$$\begin{aligned}\partial_\zeta H_f(s, t) &= \frac{1}{2}[F_f(v''_f + it) - (v''_f + it) + 1 + s(F'_f(v''_f + it) - 1)], \\ \partial_{\bar{\zeta}} H_f(s, t) &= \frac{1}{2}[F_f(v''_f + it) - (v''_f + it) - 1 - s(F'_f(v''_f + it) - 1)].\end{aligned}$$

Therefore, as a consequence of the estimates in Lemma 4.12-2,

$$\begin{aligned}& - \text{ if } \text{Im } H_f(\zeta) > 0, \\ (14) \quad & |\partial_{\bar{\zeta}} H_f(\zeta)|, |\partial_\zeta H_f(\zeta) - 1| \leq K_9 |\tau_f(H_f(\zeta))|, \\ & - \text{ if } \text{Im } H_f(\zeta) < 0,\end{aligned}$$

$$(15) \quad |\partial_{\bar{\zeta}} H_f(\zeta) + 1 + \frac{1}{2\pi\alpha(f)\mathbf{i}} \log f'(\sigma_f)|, |\partial_\zeta H_f(\zeta) - 1| \leq K_9 |\tau_f(H_f(\zeta)) - \sigma_f|.$$

The complex dilatation of H_f is given by

$$\mu_f(\zeta) := \frac{\partial_{\bar{\zeta}} H_f}{\partial_\zeta H_f}(\zeta).$$

Now consider

$$G_f := L_f^{-1} \circ H_f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{C}.$$

Since $G_f(\zeta + 1) = G_f(\zeta) + 1$ on the boundary of the domain of G_f , we can extend this map to a quasi-conformal mapping of the complex plane using this relation. By the definition, G_f maps $n \in \mathbb{Z}$ to $s + n$. Note that the complex dilatation of G_f at ζ , $\mu(G_f)(\zeta)$, is equal to $\mu_f(\zeta)$. See Figure 6.

The asymptotic translation factor of every G_f near $+i\infty$ and $-i\infty$ is finite, with a bound given in the next statement.

Sublemma 4.16. — *There exists a constant A_3 such that for all $f \in \mathcal{QLS}$, and ζ with $\operatorname{Im} \zeta > 0$*

$$|\operatorname{Im} G_f(\zeta) - \operatorname{Im} \zeta| \leq A_3(1 - \log \alpha(f))$$

Proof. — Let $\Pi := [0, 1] \times [0, D]$, and see that

$$\oint_{\partial\Pi} G_f(\zeta) d\zeta = D\mathbf{i} - \int_0^1 G_f(D\mathbf{i} + t) dt + \int_0^1 G_f(t) dt,$$

and

$$\begin{aligned} \left| \iint_{\Pi} \partial_{\bar{\zeta}} G_f(\zeta) d\zeta d\bar{\zeta} \right| &\leq \sup_{\Pi} |(L_f^{-1})'| \iint_{\Pi} |\partial_{\bar{\zeta}} H_f(\zeta)| d\zeta d\bar{\zeta} \\ &\leq A_4 \iint_{\Pi} |\tau_f(\zeta)| d\zeta d\bar{\zeta} \quad (\text{Eq. (14)}) \\ &\leq A_5 \left(1 + \iint_{[0,1] \times [1,\infty]} \left| \frac{\sigma_f}{1 - e^{-2\pi\alpha(f)w\mathbf{i}}} \right| dw d\bar{w} \right) \quad (\text{Thm. 4.6}) \\ &\leq A_5 \left(1 + \frac{2\pi\alpha(f)}{u_f(0)} \int_1^\infty \frac{1}{e^{2\pi\alpha(f)t} - 1} dt \right) \\ &\leq A_5 \left(1 + \frac{1}{u_f(0)} (\log(1 - e^{-2\pi\alpha(f)t}) \Big|_1^\infty) \right) \\ &\leq A_6(1 - \log \alpha). \end{aligned}$$

Moreover,

$$\left| \int_0^1 G_f(D\mathbf{i} + t) dt - G_f(D\mathbf{i}) \right| \text{ and } \left| \int_0^1 G_f(t) dt \right|$$

are uniformly bounded from above, independent of f . However, by Green's Formula, the integrals over $\partial\Pi$ and Π are equal. This finishes the proof of the sub-lemma. \square

By the pre-compactness of class \mathcal{IS} , $\{v_f''; f \in \mathcal{QLS}\}$ is compactly contained in $\mathbb{C} \setminus \{0\}$. Therefore, from Equation (13) and the above sub-lemma, we conclude that there exists a constant K_{13} such that for all $f \in \mathcal{QLS}$ we have

$$(16) \quad |\ell_f| \leq K_{13}(1 - \log \alpha(f)).$$

To compare the asymptotic translations of the maps G_g and G_h near infinity, we consider the map $\Omega := G_h \circ G_g^{-1}$. By the composition rule, the complex dilatation of Ω at $\xi = G_g(\zeta)$ is given by

$$\mu_\Omega(\xi) = \frac{\mu_h(\zeta) - \mu_g(\zeta)}{1 - \mu_g(\zeta)\overline{\mu_h(\zeta)}} \left(\frac{\partial_\zeta G_g(\zeta)}{|\partial_\zeta G_g(\zeta)|} \right)^2.$$

By Lemma 4.13 and Sub-lemma 4.16, when $\operatorname{Im} L_g(\xi) > 0$

$$|\mu_\Omega(\xi)| \leq A_7 \operatorname{d}_{\text{Teich}}(\pi(g), \pi(h)) |\tau_g(L_g(\xi))|,$$

and, with the help of holomorphic fixed-point formula, when $\operatorname{Im} L_g(\xi) < 0$

$$|\mu_\Omega(\xi)| \leq A_7 \operatorname{d}_{\text{Teich}}(\pi(g), \pi(h)) |\tau_g(L_g(\xi)) - \sigma_g|.$$

Let γ_1 and γ_2 be two closed horizontal curves on \mathbb{C}/\mathbb{Z} at heights $-R$ and $+R$, respectively. Also, let γ_3 be a circle of radius ε around $[0] \in \mathbb{C}/\mathbb{Z}$. All these curves are

considered with induced positive orientation. Also, let $U_{R,\varepsilon}$ be the set on the cylinder bounded by these curves.

By Green's formula,

$$(17) \quad \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} (\Omega(\xi) - \xi) \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} d\xi = \iint_{U_{R,\varepsilon}} \partial_{\bar{\xi}} \Omega(\xi) \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} d\bar{\xi} d\xi$$

As $R \rightarrow +\infty$ and $\varepsilon \rightarrow 0$, the left hand integral tends to

$$(18) \quad \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} (\Omega(\xi) - \xi) \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} d\xi \rightarrow \lim_{\text{Im } \xi \rightarrow +\infty} (\Omega(\xi) - \xi).$$

On the other hand,

$$(19) \quad \begin{aligned} & \left| \iint_{\mathbb{C}/\mathbb{Z}} \partial_{\bar{\xi}} \Omega(\xi) \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} d\bar{\xi} d\xi \right| \\ & \leq A_8 \iint_{\mathbb{C}/\mathbb{Z}} |\mu_{\Omega}(\xi)| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \\ & \leq A_8 A_7 \text{d}_{\text{Teich}}(\pi(g), \pi(h)) \left(\iint_{L_g(\mathbb{C}/\mathbb{Z})^+} |\tau_g(H_g(\xi))| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \right. \\ & \quad \left. + \iint_{L_g(\mathbb{C}/\mathbb{Z})^-} |\tau_g(H_g(\xi)) - \sigma_g| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \right) \\ & \leq A_9 \text{d}_{\text{Teich}}(\pi(g), \pi(h)) \frac{1}{\alpha(g)}, \end{aligned}$$

by estimating the definite integrals.

Finally,

$$\begin{aligned} & |\alpha(g)\ell_g - \alpha(h)\ell_h| \\ & = \alpha(g) \lim_{\text{Im } \zeta \rightarrow +\infty} |(H_g \circ G_g^{-1}(\xi) - \xi) - (H_h \circ G_h^{-1}(\xi) - \xi)| \\ & = \alpha(g) \lim_{\text{Im } \zeta \rightarrow +\infty} |\text{Im } v_g'' - \text{Im } v_h'' - \text{Im}(G_g(\zeta) - \zeta) + \text{Im}(G_h(\zeta) - \zeta)| \quad (\text{Eq. 13}) \\ & = \alpha(g) |\text{Im } v_g'' - \text{Im } v_h''| + \alpha(g) \lim_{\text{Im } \zeta \rightarrow +\infty} |\text{Im}(G_g(\zeta) - G_h(\zeta))| \quad (\text{Eq. 12}) \\ & \leq (A_8 + A_9) \text{d}_{\text{Teich}}(\pi(g), \pi(h)) \quad (\text{Eq. 19}) \end{aligned}$$

□

Proof of Lemma 4.9. — In what follows, all the constants A_1, A_2, A_3, \dots , depending only on the class \mathcal{QIS} .

Let $f_\alpha := f_0(e^{2\pi\alpha\mathbf{i}}z)$ be in \mathcal{QIS} with $f'_\alpha(0) = e^{2\pi\alpha\mathbf{i}}$, and let F_α denote the lift of f_α under τ_{f_α} . Also, let $L_\alpha := L_{f_\alpha}$ denote the linearizing map of F_α with asymptotic translation $\ell_\alpha := \ell_{f_\alpha}$ near infinity. By Inequality (16),

$$(20) \quad |\ell_\alpha| \leq C - C \log \alpha.$$

We continue to use the notations in the proof of Lemma 4.8. In what follows, all the constants A_1, A_2, \dots depend only on the class \mathcal{IS} .

We want to show that there is a constant A_1 with

$$(21) \quad \left| \alpha \frac{d}{d\alpha} \ell_\alpha \right| \leq A_1.$$

First note that

$$\left| \frac{dv''_\alpha}{d\alpha} \right| \leq \left| \frac{d}{d\alpha} \left(\frac{1}{2\pi\alpha} \log \left(1 - \frac{1 - e^{2\pi\alpha i}}{f_\alpha^{os}(-4/27)u_{f_\alpha}(0)} \right) \right) \right| \leq A_2.$$

Let $H_\alpha := H_{f_\alpha}$ defined in the proof of Lemma 4.8, $G_\alpha := G_{f_\alpha}$, and $\mu_\alpha := \mu_{H_\alpha} = \mu_{G_\alpha}$. Fix $\alpha, \beta \in (0, \alpha^*]$ close together, and define $\Omega := G_\beta \circ G_\alpha^{-1}$. It follows from Lemma 4.14 that $|d\mu_\alpha/d\alpha|$ is either less than $A_3|\tau_\alpha(H_\alpha(\zeta))|$ or $A_3|\tau_\alpha(H_\alpha(\zeta)) - \sigma_\alpha|$, depending on the height of $H_\alpha(\zeta)$. and hence

$$(22) \quad |\mu_\Omega(\xi)| \leq A_4|\beta - \alpha||\tau_\alpha(L_\alpha(\xi))|, \text{ when } \operatorname{Im} L_\alpha(\xi) > 0$$

$$(23) \quad |\mu_\Omega(\xi)| \leq A_4|\beta - \alpha||\tau_\alpha(L_\alpha(\xi)) - \sigma_\alpha|, \text{ when } \operatorname{Im} L_\alpha(\xi) < 0$$

We have

$$\begin{aligned} & \left| \iint_{\mathbb{C}/\mathbb{Z}} \partial_{\bar{\xi}} \Omega(\xi) \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} d\bar{\xi} d\xi \right| \\ & \leq A_5 \iint_{\mathbb{C}/\mathbb{Z}} |\mu_\Omega(\xi)| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \\ (24) \quad & \leq A_5 A_4 |\beta - \alpha| \left(\iint_{L_\alpha(\mathbb{C}/\mathbb{Z})^+} |\tau_g(H_g(\xi))| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \right. \\ & \quad \left. + \iint_{L_\alpha(\mathbb{C}/\mathbb{Z})^-} |\tau_g(H_g(\xi)) - \sigma_g| \left| \frac{\tan(\pi\xi) + \mathbf{i}}{2 \tan(\pi\xi)} \right| d\bar{\xi} d\xi \right) \\ & \leq A_6 |\beta - \alpha| \frac{1}{\alpha(f)} \end{aligned}$$

Hence, combining with equations (17) and (18),

$$\begin{aligned} \left| \frac{d\ell_\alpha}{d\alpha}(\alpha(f)) \right| &= \left| \lim_{\beta \rightarrow \alpha(f)} \frac{v''_{\alpha(f)} - v''_\beta + \lim_{\xi \rightarrow \infty} (\Omega(\xi) - \xi)}{\alpha(f) - \beta} \right| \\ &\leq A_2 + A_6 \frac{1}{\alpha(f)} \leq \frac{A_2 + A_6}{\alpha(f)}, \end{aligned}$$

proving the desired Inequality (21).

Combining the two inequalities (20) and (21) we conclude that

$$\left| \frac{d(\alpha \ell_\alpha)}{d\alpha} \right| \leq C + A_2 + A_5 - C \log \alpha,$$

which can be integrated to produce the inequality in the lemma. \square

4.5. Univalent maps with small Schwarzian derivative. — In this subsection we prove Proposition 4.2. Given $f, g \in QIS$, set $h(z) := f(e^{2\pi(\alpha(g) - \alpha(f))\mathbf{i}} z)$ and note that by virtue of Theorem 3.1 we only need to show that

$$d_{\text{Teich}}(\pi \circ \mathcal{R}(f), \pi \circ \mathcal{R}(h)) \leq K_2 |\alpha(f) - \alpha(h)|,$$

for some constant K_2 depending only on \mathcal{QLS} . We shall prove the above inequality in several steps, here.

Within this subsection all the constants A_1, A_2, A_3, \dots depend only on \mathcal{QLS} , unless otherwise stated. For our convenience, we introduce the following notations.

Given $f_0 \in \mathcal{IS} \cup \{z \mapsto z + z^2\}$ and $\alpha \in (0, \alpha^*]$, define

$$f_\alpha(z) := f_0(e^{2\pi\alpha i}z).$$

We write the renormalized map in the form

$$(25) \quad \mathcal{R}(f_\alpha)(w) = P \circ \psi_\alpha^{-1}(e^{2\pi\frac{1}{\alpha}i}w), \text{ with } \psi_\alpha : V \rightarrow \mathbb{C}.$$

The map $\psi_\alpha : V \rightarrow V_\alpha := \psi_\alpha(V)$ is univalent, $\psi_\alpha(0) = 0$, and $\psi'_\alpha(0) = 1$. Recall that by Theorem 3.1 the map ψ_α has univalent extension onto the larger domain U .

Lemma 4.17. — *There exists a constant A_1 , depending only on \mathcal{QLS} , and a Jordan domain W with $\partial W \subset U \setminus \overline{V}$ such that for all $z \in W$ and $\alpha \in (0, \alpha^*]$*

$$|\frac{\partial \psi_\alpha}{\partial \alpha}(z)| \leq A_1.$$

Proof. — Let $\Phi_\alpha : \mathcal{P}_\alpha \rightarrow \mathbb{C}$ denote the normalized Fatou coordinate of f_α that can be decomposed as in $\Phi_\alpha^{-1} = \tau_\alpha \circ L_\alpha$, where τ_α is the covering map defined in Equation (8) for f_α . The lift of f_α under τ_α is denoted by F_α . Let $\mathcal{C}_\alpha \cup \mathcal{C}_\alpha^\sharp$ denote the sets defined in Equation (3) for the map f_α . Also, let $k_\alpha := k_{f_\alpha}$ denote the smallest positive integer for which $S_\alpha := \mathcal{C}_\beta^{-k_\alpha} \cup (\mathcal{C}_\alpha^\sharp)^{-k_\alpha}$ is contained in

$$\{z \in \mathcal{P}_\alpha \mid \operatorname{Re} \Phi_\alpha(z) \in (1/2, \lfloor \alpha^{-1} \rfloor - \mathbf{k} - 1/2)\}.$$

Consider

$$E_\alpha : \Phi_\alpha(S_\alpha) \rightarrow \Phi_\alpha(\mathcal{C}_\alpha \cup \mathcal{C}_\alpha^\sharp); E_\alpha(w) := \Phi_\alpha \circ f_\alpha^{k_\alpha} \circ \Phi_\alpha^{-1}(w).$$

By the definition of renormalization, E_α projects under \mathbb{Exp} to give us $\mathcal{R}(f_\alpha)$. It is also convenient to consider the conjugate map

$$(26) \quad \tilde{E}_\alpha : L_\alpha(\Phi_\alpha(S_\alpha)) \rightarrow L_\alpha(\Phi_\alpha(\mathcal{C}_\alpha \cup \mathcal{C}_\alpha^\sharp)); \tilde{E}_\alpha := L_\alpha \circ E_\alpha \circ L_\alpha^{-1}.$$

However, by the definition of F_α , we also have

$$(27) \quad \tilde{E}_\alpha = F_\alpha^{k_\alpha} - \frac{1}{\alpha}.$$

By pre-compactness of the class \mathcal{QLS} and the uniform bound in Lemma 4.10, there exists a neighborhood X_α of $L_\alpha(\Phi_\alpha(S_\alpha))$, of a size depending only on \mathcal{QLS} , such that \tilde{E}_α is defined on X_α . Hence, by Lemma 4.15, there exists a neighborhood Z_α of radius A_2 about $\Phi_\alpha(S_\alpha)$ such that $L_\alpha(Z_\alpha) \subseteq X_\alpha$.

On the other hand, since each ψ_α has univalent extension onto U , by Kőbe distortion Theorem, there exists a constant A_3 depending only on V and the conformal modulus of $U \setminus V$ such that $\partial(\psi_\alpha(V)) \subseteq B(0, A_3) \setminus B(0, 1/A_3)$. Hence,

$$\forall w \in \mathbb{Exp}^{-1}(\partial V_\alpha \cdot e^{-2\pi\frac{1}{\alpha}i}), \quad |\operatorname{Im} w| \leq A_4.$$

In particular, a period of $\text{Exp}^{-1}(\partial V_\alpha \cdot e^{-2\pi \frac{1}{\alpha} \mathbf{i}})$ lies on the boundary of $\text{Dom } E_\alpha$. This implies that there exists a domain W with $\partial W \subseteq U \setminus V$, such that

$$\psi_\alpha(W) \cdot e^{-2\pi \frac{1}{\alpha} \mathbf{i}} \subseteq \text{Exp}(Z_\alpha).$$

Fix $\alpha \in (0, \alpha^*]$ and note that by the continuity of Φ_α in terms of α , $k_\beta = k_\alpha$, for $\beta \in (0, \alpha^*]$ sufficiently close to α . Differentiating Equation (27) with respect to α and using Lemmas 4.14 and 4.10, one obtains

$$\forall w \in X_\alpha, \left| \frac{\partial \tilde{E}_\alpha}{\partial \alpha}(w) \right| \leq A_5 + \frac{1}{\alpha^2}.$$

From the proof of Lemma 4.9, there exists a constant A_4 depending only on $A_2 + A_4$ such that

$$\forall \beta \in (0, \alpha^*], \forall w \in \mathbb{C}, \text{ with } |\text{Im } w| \leq A_2 + A_4, \left| \frac{\partial L_\beta}{\partial \beta}(w) \right| \leq A_6.$$

Briefly speaking, because of Lemma 4.15, it is enough to bound $L_\beta^{-1} \circ L_\alpha$ by a constant times $|\beta - \alpha|$. By virtue of the explicit estimates on the model map H_{f_α} considered in the proof of Lemma 4.9, it suffices to bound $G_{f_\beta} \circ G_{f_\alpha}^{-1}$. But the dilatation of this map is bounded by a universal constant times $|\beta - \alpha|$, as in Equation (22). From the singular integral representation of the solution of Beltrami Equation, see [Leh87], one can see that $|G_{f_\beta} \circ G_{f_\alpha}^{-1}(z) - z|$ is bounded by a constant times $|\beta - \alpha|$.

One concludes from the above inequality and the one in Lemma 4.15, by differentiating Equation (26), the inequality

$$\forall w \in Z_\alpha, \text{ with } |\text{Im } w| \leq A_2 + A_4, \left| \frac{\partial E_\alpha}{\partial \alpha}(w) \right| \leq A_7 \frac{1}{\alpha^2}.$$

Projecting onto a neighborhood of 0, since $|\text{Exp}'(w)|$ is uniformly bounded from above and below on $\{w \in \mathbb{C}; |\text{Im } w| \leq A_2 + A_4\}$, we obtain

$$\forall z \in \psi_\alpha(W) \cdot e^{-2\pi \frac{1}{\alpha} \mathbf{i}}, \left| \frac{\partial \mathcal{R}(f_\alpha)}{\partial \alpha}(z) \right| \leq A_8 \frac{1}{\alpha^2}$$

The absolute value of P' is uniformly bounded from above and below on $W \setminus V$. Now, by differentiating (25) with respect to α and using the above estimate, we obtain the estimate in the lemma at points in $W \setminus V$. Since $\partial \psi_\alpha / \partial \alpha$ is holomorphic in z , for each fixed α , by the maximum principle, we only need to prove that the uniform bound in the lemma holds on $W \setminus V$. \square

Fix $\alpha \in (0, \alpha^*]$ and for $\beta \in (0, \alpha^*]$ define the univalent map

$$\Omega_\beta := \psi_\beta \circ \psi_\alpha^{-1} : V_\alpha \rightarrow V_\beta.$$

The *Schwarzian derivative* of Ω_β on V_α is defined as

$$S \Omega_\beta := \left(\frac{\Omega_\beta''}{\Omega_\beta'} \right)' - \frac{1}{2} \left(\frac{\Omega_\beta''}{\Omega_\beta'} \right)^2.$$

Since each Ω_β has univalent extension onto $U_\alpha := \psi_\alpha(U)$, the above quantity is uniformly bounded independent of β and f_0 , by Koebe distortion Theorem. The value of this derivative measures the deviation of Ω_β from the Möbius transformations at

points in U_α . Let η_α denote the Poincaré density of V_α , that is, $\eta|dz|$ is a complete metric of constant -1 curvature on V_α . The hyperbolic sup-norm of $S\Omega_\beta$ on V_α is defined as

$$\|S\Omega_\beta\|_{V_\alpha} := \sup_{z \in V_\alpha} |S\Omega_\beta(z)|\eta_\alpha(z)^{-2}.$$

Lemma 4.18. — *There exists a constant A_9 , depending only on A_1 and the conformal modulus of $W \setminus V$, such that for all $\alpha, \beta \in (0, \alpha^*]$ we have*

$$\|S\Omega_\beta\|_{V_\alpha} \leq A_9|\beta - \alpha|.$$

Proof. — By the definition of the conformal modulus, there exists a positive constant r , depending only on the diameter of V and the modulus of $W \setminus V$ such that for all $z \in V$, $B(z, r) \subseteq W$. Then, it follows from the estimate in Lemma 4.17, and Cauchy Integral Formulas for the derivatives that on the set V ,

$$\left| \frac{\partial(\psi_\alpha - \psi_\beta)}{\partial z} \right|, \left| \frac{\partial^2(\psi_\alpha - \psi_\beta)}{\partial z^2} \right|, \left| \frac{\partial^3(\psi_\alpha - \psi_\beta)}{\partial z^3} \right| \leq A_{10}|\alpha - \beta|.$$

On the other hand, from the definition of Schwarzian derivative (or see [Leh76], Section II.1.3),

$$\|S\Omega_\beta\|_{V_\alpha} = \|S\psi_\alpha - S\psi_\beta\|_V.$$

Combining the above estimates one obtains the desired inequality. \square

It is a classical result in Teichmüller Theory, see [Ahl63] or [Leh76, Theorem 4.1], that there exists a constant A_{11} , depending only on V , such that $\Omega_\beta : V_\alpha \rightarrow V_\beta$ has a quasi-conformal extension onto \mathbb{C} with complex dilatation less than $A_{11}\|S\Omega_\beta\|_{V_\alpha}$. By the definition of the Teichmüller distance, we obtain the following corollary that finishes the proof of the proposition.

Corollary 4.19. — *There exists a constant A_{12} , depending only on \mathcal{QIS} , such that for all $\alpha, \beta \in (0, \alpha^*]$, we have*

$$d_{\text{Teich}}(\pi \circ \mathcal{R}(f_\alpha), \pi \circ \mathcal{R}(f_\beta)) \leq A_{12}|\alpha - \beta|.$$

4.6. Upper bound on the size of Siegel disks. —

Proof of Proposition 4.3. — We shall show that there exists a constant A_{13} such that for every $f \in \mathcal{QIS}$ there exists a sequence of points z_n , $n = 0, 1, 2, \dots$ in the forward orbit of the unique critical value of f satisfying the inequality

$$\log d(0, z_n) \leq A_{13} + \sum_{i=0}^n \beta_{i-1} \log \alpha_i.$$

Assuming this for a moment, if z is an accumulation point of this sequence, it must lie outside of the Siegel disk of f and that $\log d(0, z) \leq A_{13} - Y(\alpha(f))$. By 1/4-Theorem, we obtain the desired inequality $\log r(f) + Y(\alpha(f)) \leq A_{13} + \log 4$.

For $n \geq 1$, let $\zeta_n := \lfloor (1/\alpha_n - \mathbf{k})/2 \rfloor$, and then inductively define the sequence $\zeta_{n-1}, \zeta_{n-2}, \dots, \zeta_0$ so that for $i = 0, 1, 2, \dots, n-1$,

$$|\operatorname{Re}(\zeta_i) - (1/\alpha_i - \mathbf{k})/2| \leq 1/2, \text{ and } \mathbb{E}xp(\zeta_i) = \Phi_{i+1}^{-1}(\zeta_{i+1}),$$

where, $\mathbb{E}xp$ is defined in Equation (5) and Φ_i is the perturbed Fatou coordinate of the map $f_i := \mathcal{R}^i(f)$. Then, define $z_n := \Phi_0^{-1}(\zeta_0)$.

By an inductive argument we prove that each $\Phi_i^{-1}(\zeta_i)$ is in the forward orbit of the critical value of f_i , $-4/27$. For $i = n$, it is clear; $\Phi_n(-4/27) = 1$, and therefore, by the equivariance property of Φ_n , $\Phi_n^{-1}(\zeta_n)$ is in the forward orbit of $-4/27$. Choose the smallest positive integer t_n with $f_n^{\circ t_n}(-4/27) = \Phi_n^{-1}(\zeta_n)$.

To prove the statement for $i = n - 1$, first note that by the definition of the renormalization there are $x_i \in \Phi_{n-1}(S_{f_{n-1}})$ and $y_i \in \Phi_{n-1}(\mathcal{C}_{f_{n-1}} \cup \mathcal{C}_{f_{n-1}}^\#)$, for $i = 0, 1, 2, \dots, t_n - 1$ such that

$$\begin{aligned} \mathbb{E}xp(x_i) &= f_n^{\circ i}(-4/27), \quad \mathbb{E}xp(y_i) = f_n^{\circ i+1}(-4/27), \\ \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(x_i) &= y_i. \end{aligned}$$

Moreover, choose integers s_i , so that $x_0 = s_0$, $y_{i-1} + s_i = x_i$, for $i = 0, 1, 2, \dots, t_n - 1$, and $\zeta_{n-1} = y_{t_n-1} + s_{t_n}$. This implies that iterating $-4/27$, $s_0 + s_1 + \dots + s_{t_n} + t_n k_{n-1}$ number of times, we reach $\Phi_{n-1}^{-1}(\zeta_{n-1})$. One continues this process to reach level zero.

Next, we estimate the size of z_n . To do this we first show that there exists a constant A_{14} , depending only on \mathcal{QLS} , such that for all $i = n, n - 1, \dots, 0$

$$(28) \quad \text{Im } \zeta_{i-1} \geq \alpha_i \text{Im } \zeta_i - \log \alpha_i - A_{14}.$$

Recall the maps H_f and G_f defined in the proof of Lemma 4.8, and choose an integer t_i such that $\zeta_i + t_i \in G_{f_i}([0, 1] \times \mathbb{R})$. We have $L_{f_i}(\zeta_i) = F_{f_i}^{\circ -t_j} \circ H_{f_i} \circ G_{f_i}^{-1} \zeta_i + t_i$. Then, it follows from Sublemma 4.16 and the estimate in Lemma 4.12-2 that

$$|L_{f_i}(\zeta_i) - \zeta_i| \leq A_{15}(1 - \log \alpha_i).$$

Inequality (28) results from an easy calculation of the explicit formula $\mathbb{E}xp^{-1} \circ \tau_{f_i}$.

Recursively putting Equation (28) together, one obtains the desired inequality on the size of z_n . \square

Proof of Proposition 4.4. — By the definition of $C(f)$, Proposition 4.3, Equation (16) and Equation (9), for positive α we have

$$\lim_{\alpha \rightarrow 0^+} \Upsilon(z \mapsto f_0(e^{2\pi\alpha i} z)) = \lim_{\alpha \rightarrow 0^+} C(z \mapsto f_0(e^{2\pi\alpha i} z)) = -\log |f''(0)| + \log(4\pi).$$

When α is negative, one considers the conjugate map $s \circ (z \mapsto f_0(e^{2\pi\alpha i} z)) \circ s$, where $s(z) = \bar{z}$, and obtains a family of the form $z \mapsto g_0(e^{-2\pi\alpha i} z)$. Moreover, the class \mathcal{IS} is closed under this conjugation, $|f_0''(0)| = |g_0''(0)|$, and the conformal radius of the Siegel disk is preserved under this conjugation. Hence, the equality for the left limit follows from the above one. \square

5. Hölder

5.1. Introduction. — We continue to use the Riemannian metric $ds = |\log |x|| |dx|$ on the interval $[-1/2, 1/2]$. The distance with respect to this metric between two points $x, y \in [-1/2, 1/2]$ will be denoted $d_{\log}(x, y)$. The length of $[-1/2, 1/2]$ for this metric is finite (it is equal to $1 + \log 2$). This distance is Hölder continuous with respect to the Euclidean distance, for any exponent in $(0, 1)$ (see Section 5.6.2).

Recall our notations: \mathcal{IS} is (a big enough subset of) the Inou-Shishikura class. Maps $f \in \mathcal{IS}$ have rotation number 0: $f(0) = 0$ and $f'(0) = 1$. Given a holomorphic map f whose domain contains 0, fixing 0, and with $f'(0) \neq 0$, we denote

$$f = f_0 \circ R_{\alpha(f)}$$

with $f'_0(0) = 1$ and

$$R_\theta(z) = e^{2\pi i \theta} z.$$

With the notation of Sections 3 and 4, $f_0 = \pi(f)$. For $A \subset \mathbb{R}$, \mathcal{IS}_A is the set of maps f with $f_0 \in \mathcal{IS}$ and $\alpha(f) \in A$. In Section 3, we also introduced the family of quadratic polynomials Q_α that is closely related to P_α :

$$P_\alpha = A_\alpha \circ Q_\alpha \circ A_\alpha^{-1}, \quad A_\alpha(z) = (e^{2\pi i \alpha})^2 \frac{27}{16} z.$$

In particular, with $r(f)$ denoting the conformal radius of the Siegel disk of f :

$$r(P_\alpha) = \frac{27}{16} r(Q_\alpha)$$

The renormalization $\mathcal{R}(f)$ was defined in Section 3 for $\alpha(f)$ small and positive, more precisely for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \alpha^*]$. We prefer here to work with P_α instead of Q_α , so we define

$$\mathcal{R}(P_\alpha) = \mathcal{R}(Q_\alpha).$$

To extend the renormalization to maps with $\alpha \in [-\alpha^*, 0)$ we proceed as follows: let $s(z) = \bar{z}$; let us shorten $s \circ f \circ s^{-1}$ as $sf s^{-1}$ then $\alpha(sf s^{-1}) = -\alpha(f) \in (0, \alpha^*]$; for $f \in \mathcal{IS}_\mathbb{R}$ with $\alpha(f) \in (-1/2, 0) \cup (0, 1/2)$, let

$$\begin{aligned} \mathcal{S}(f) &= f \text{ if } \alpha(f) \in (0, 1/2) \text{ and} \\ \mathcal{S}(f) &= sf s^{-1} \text{ if } \alpha(f) \in (-1/2, 0). \end{aligned}$$

The symbol \mathcal{S} stands for *saw*, because it acts on the rotation number as the saw map mentionned in Section 2.1. For $\alpha \in (-1/2, 1/2)$, let also

$$\mathcal{S}(Q_\alpha) = Q_{|\alpha|}, \quad \mathcal{S}(P_\alpha) = P_{|\alpha|}.$$

Then the composition of \mathcal{RS} is an extension of \mathcal{R} to maps with $\alpha(f) \in [-\alpha^*, 0) \cup (0, \alpha^*]$. Let us denote $\mathcal{RS}_0(f) = (\mathcal{RS}(f))_0$, so that whenever $\alpha(f) \in [-\alpha^*, 0) \cup (0, \alpha^*]$:

$$\mathcal{RS}(f) = \mathcal{RS}_0(f) \circ R_{1/|\alpha(f)|}.$$

Note also that

$$Y(-\alpha) = Y(\alpha), \quad r(sf s^{-1}) = r(f), \quad \Upsilon(sf s^{-1}) = \Upsilon(f)$$

thus

$$r(\mathcal{S}f) = r(f), \quad \Upsilon(\mathcal{S}f) = \Upsilon(f),$$

and similarly

$$r(P_{-\alpha}) = r(P_\alpha), \quad \Upsilon(P_{-\alpha}) = \Upsilon(P_\alpha), \quad r(SP_\alpha) = r(P_\alpha), \quad \Upsilon(SP_\alpha) = \Upsilon(P_\alpha).$$

Recall that the set HT_N is the set of irrational numbers whose modified continued fraction entries are all $\geq N$, and that $\mathcal{IS}_N := \mathcal{IS}_{\text{HT}_N}$. Note that for $N \geq 1/\alpha^*$,

$\text{HT}_N \subset A$ with $A = [-\alpha^*, 0) \cup (0, \alpha^*]$, and that maps f in \mathcal{IS}_N or in $\{P_\alpha \mid \alpha \in \text{HT}_N\}$ are infinitely renormalizable for \mathcal{RS} .

Assume a distance function $d_T(f_0, g_0)$ has been defined⁽⁶⁾ over \mathcal{IS}_0 , has been extended to $\mathcal{IS}_{\mathbb{R}}$ as follows:

$$d_{\log}(f, g) = d_{\log}(\alpha(f), \alpha(g)) + d_T(f_0, g_0)$$

and that the following holds:

$$f \in \mathcal{IS}_N \mapsto C(f) = \Upsilon(f) - |\alpha(f)|\Upsilon(\mathcal{RS}(f))$$

is bounded and is Lipschitzian over $\mathcal{IS}_N \cap [-1/2, 1/2]$ with respect to the metric d_{\log} ,

$$\alpha \in \text{HT}_N \mapsto C(P_\alpha) = \Upsilon(P_\alpha) - |\alpha|\Upsilon(\mathcal{RS}(P_\alpha))$$

too. Note that

$$C(\mathcal{S}f) = C(f) \text{ and } C(\mathcal{S}P_\alpha) = C(P_\alpha).$$

Assume that the transversal part of renormalization is Lipschitzian over \mathcal{IS}_N with respect to the metric d_{\log} , and transversally contracting (see below). Assume also that Υ is bounded over \mathcal{IS}_N (in [BC04] it has been proved that Υ is bounded over $\mathcal{P}_{\mathbb{R}}$). This sums up as follows:

. $\forall f, g \in \mathcal{IS}_N$:

1. $\Upsilon(f) \leq K_0$
2. $|C(f) - C(g)| \leq K_1 d_{\log}(f, g)$
3. $|C(f)| \leq K_3$
4. If $\alpha(f)$ and $\alpha(g)$ have the same sign, then
 $d_T(\mathcal{RS}_0(f), \mathcal{RS}_0(g)) \leq \lambda d_T(f_0, g_0) + K_2 |\alpha(f) - \alpha(g)|$
5. $\lambda < 1$

. $\forall \alpha, \beta \in (-1/N, 0) \cup (0, 1/N)$:

6. $|C(P_\alpha) - C(P_\beta)| \leq K_1 d_{\log}(\alpha, \beta)$
7. $|C(P_\alpha)| \leq K_3$
8. If α and β have the same sign, then
 $d_T(\mathcal{RS}_0(P_\alpha), \mathcal{RS}_0(P_\beta)) \leq K_2 |\alpha - \beta|$

Under the above hypotheses, we will prove:

Theorem 5.1. — $\exists A, B > 0$ such that $\forall f, g \in \mathcal{IS}_N$,

$$|\Upsilon(f) - \Upsilon(g)| \leq A d(\alpha(f), \alpha(g))^{1/2} + B d(f_0, g_0)$$

and $\forall \alpha, \beta \in \text{HT}_N$,

$$|\Upsilon(P_\alpha) - \Upsilon(P_\beta)| \leq A d(\alpha(f), \alpha(g))^{1/2}.$$

Remark. — As previously noted, the (uniform) modulus of continuity of Υ with respect to the rotation number will not be better than $1/2$ -Hölder. However, the pointwise modulus of continuity can be better at some values. Note that since Hölder continuous maps are uniformly continuous, Υ extends to the closure of HT_N , which

⁽⁶⁾The subscript T stands for Teichmüller, but our proof is valid for any metric with the required properties stated here.

consists of HT_N plus some rationals. Let x be one of them (for instance, $x = 0$). Then with a finite number of renormalizations, it follows from the continuity of Υ , and the above hypotheses that Υ has a modulus of continuity at $x \in \mathbb{Q}$ of the form:

$$|\Upsilon(x + \epsilon) - \Upsilon(x)| \leq c_x |\epsilon \log |\epsilon||$$

(in a neighborhood of x). This is much better than $|\epsilon|^{1/2}$, almost as good as $|\epsilon|$. However, c_x depends on x . It cannot be better than $|\epsilon \log |\epsilon||$, though: from [Ché08], we know that, for quadratic polynomials, at rationals x and for some well chosen subsequences $\epsilon_n \rightarrow 0$, the function Υ has expansion of the form $\Upsilon(x + \epsilon_n) = a_x + b_x \epsilon_n \log |\epsilon_n| + c_x \epsilon_n + o(\epsilon_n)$.

Remark. — Theorem 5.1 could be proved under weaker hypotheses. Indeed, the term $d_{\log}(\alpha(f), \alpha(g))$ in Hyp. 2, $d_{\log}(\alpha, \beta)$ in Hyp. 6, $|\alpha(f) - \alpha(g)|$ in Hyp. 4 and $|\alpha - \beta|$ in Hyp. 8 could respectively be replaced by $|\alpha(f) - \alpha(g)|^\delta$, $|\alpha - \beta|^\delta$, $|\alpha(f) - \alpha(g)|^\delta$ and $|\alpha - \beta|^\delta$ with $\delta \in (1/2, 1)$: the proofs would still work.

Since $\Upsilon(P_\alpha) = \Upsilon(Q_\alpha) + \log \frac{27}{16}$ and $\mathcal{R}(P_\alpha) := \mathcal{R}(Q_\alpha)$, thus $C(P_\alpha) = C(Q_\alpha) + \log \frac{27}{16}$, Hypotheses 6, 7 and 8 are equivalent if P_α is replaced by Q_α . Hypothesis 1 has been proved in Proposition 4.3. Hypotheses 3 and 7 follow from Proposition 4.3 and the definition of $C(f)$. Hypotheses 4, 5, and 8 have been proved in Proposition 4.2. Hypotheses 2 and 6 have been proved in Proposition 4.1 in the particular case when $\alpha(f)$ and $\alpha(g)$ are both positive.

Proof of Hypotheses 2 and 6. — The case when $\alpha(f)$ and $\alpha(g)$ are both negative follows at once from the case when they are both positive since $C(f) = C(\mathcal{S}f)$. For the case when $\alpha(f)$ and $\alpha(g)$ have opposite sign, let us assume $\alpha(f) < 0 < \alpha(g)$ (the other case follows by permuting f and g). For Hypothesis 6, it is also very simple because $\mathcal{S}P_\alpha = P_{-\alpha}$ hence $|C(P_\alpha) - C(P_\beta)| = |C(P_{-\alpha}) - C(P_\beta)| \leq K_1 d_{\log}(-\alpha, \beta) < K_1 d_{\log}(\alpha, \beta)$.⁽⁷⁾ For Hypothesis 2, recall that we proved in Proposition 4.4 that $\alpha \mapsto \Upsilon(f_0 \circ R_\alpha)$ has left and right limits at 0 that are equal. Since Υ is bounded (Hypothesis 1) and $C(f) := \Upsilon(f) - |\alpha(f)|\Upsilon(\mathcal{R}\mathcal{S}(f))$, it follows that $\alpha \mapsto C(f_0 \circ R_\alpha)$ also has left and right limits at 0 that are equal. Then:

$$\begin{aligned} C(f) - C(g) &= C(f) - \lim_{\alpha \nearrow 0} C(f_0 \circ R_\alpha) \\ &\quad + \lim_{\alpha \nearrow 0} C(f_0 \circ R_\alpha) - \lim_{\alpha \searrow 0} C(f_0 \circ R_\alpha) \\ &\quad + \lim_{\alpha \searrow 0} C(f_0 \circ R_\alpha) - C(f_0 \circ R_{\alpha(g)}) \\ &\quad + C(f_0 \circ R_{\alpha(g)}) - C(g). \end{aligned}$$

Recall that $d_{\log}(f, g) := d_{\log}(\alpha(f), \alpha(g)) + d_T(f_0, g_0)$. By the case when both rotation numbers are negative, we know that for $\alpha \in (\alpha(f), 0)$, $|C(f) - C(f_0 \circ R_\alpha)| \leq$

⁽⁷⁾For maps in \mathcal{ZS}_N , we cannot use this trick because there is also the term $d_T(f_0, g_0)$ to take into account, and $d_T((sfs^{-1})_0, g_0) = d_T(sf_0s^{-1}, g_0)$ could be much bigger than $d_T(f_0, g_0)$, even when $\alpha(f)$ and $\alpha(g)$ are small.

$K_1 d_{\log}(f, f_0 \circ R_\alpha) = 0 + K_1 d_{\log}(\alpha(f), \alpha)$. Thus by passing to the limit $\alpha \xrightarrow{<} 0$, we get that the first term in the sum above is $\leq K_1 d_{\log}(\alpha(f), 0)$. Similarly, the third is $\leq K_1 d_{\log}(0, \alpha(g))$. We have proved that the second term is null. For the fourth term, by the case when both rotation numbers are positive, we get $|C(f_0 \circ R_{\alpha(g)}) - C(g)| \leq K_1 d_{\log}(f_0 \circ R_{\alpha(g)}, g) = K_1 d_T(f_0, g_0) + 0$. All in all: $|C(f) - C(g)| \leq K_1 (d_{\log}(\alpha(f), 0) + d_{\log}(0, \alpha(g)) + d_T(f_0, g_0))$. Since $\alpha(f)$ and $\alpha(g)$ have opposite signs, $d_{\log}(\alpha(f), 0) + d_{\log}(0, \alpha(g)) = d_{\log}(\alpha(f), \alpha(g))$. Whence $|C(f) - C(g)| \leq K_1 d_{\log}(f, g)$. \square

5.2. Discussion about rotation number one half. — The Marmi Moussa Yoccoz conjecture concerns a function Υ that is defined on the 1-torus \mathbb{R}/\mathbb{Z} (or it can be viewed as a 1-periodic function on \mathbb{R}). Recall the number $N \geq$ involved in the definition of high type numbers (using modified continued fractions). Note that for $N = 2$, the high type numbers are all the irrationals! In \mathbb{R}/\mathbb{Z} , the set of high type numbers is bounded away from $1/2$ as soon as $N \neq 2$. According to discussion with them, the number N provided by Inou and Shishikura is likely to be no less than 20. However, it is widely believed that a modification of their construction would work for all rotation numbers. But extra caution should be taken near rotation number $1/2$: indeed, it is known that as α varies from 0 to $1/2$, the fundamental cylinders used in parabolic renormalization tend to different domains as when α varies from 0 to $-1/2$. But in \mathbb{R}/\mathbb{Z} , $-1/2 = 1/2$. *This means that there is a discontinuity of cylinder renormalization as one crosses $1/2$.* To solve this, we believe one should use an extension of one of the two branches of the renormalization operator beyond $1/2$. However, in the present article, we will not worry with that and we will assume $N \geq 3$. We will work in \mathbb{R} instead of \mathbb{R}/\mathbb{Z} and make statements about $\text{HT}_N \cap [-1/2, 1/2]$.

Remark. — Crossing 0 also creates problems and a discontinuity of the renormalization operator, but sizes get multiplied by a factor tending to 0, which absorbs the discontinuity: this is used in the proof of hypothesis 2 in Section 5.1.

5.3. Proof, first steps. — The proof of Theorem 5.1 is split across the remaining subsections of Section 5.

For $f \in \mathcal{IS}_{\mathbb{R}}$ recall that $\mathcal{S}(f) = f$ if $\alpha(f) \in (0, 1/2) \bmod \mathbb{Z}$ and $\mathcal{S}(f) = s \circ f \circ s$ if $\alpha(f) \in (-1/2, 0) \bmod \mathbb{Z}$, where $s(z) = \bar{z}$. To shorten the notations in subsequent computations, we will denote

$$f_n = \mathcal{S}(\mathcal{RS})^n(f) \text{ and } g_n = \mathcal{S}(\mathcal{RS})^n(g).$$

They are defined so that the sequence of rotation numbers of f_n coincides with the sequence $\alpha_n(f)$ associated to the modified continued fraction of $\alpha(f)$. Note that $\mathcal{S}^2 = \text{id}$ and thus $\mathcal{S}(\mathcal{RS})^n = (\mathcal{SRS})^n$. However, it is important to note that \mathcal{S} is not continuous with respect to f as $\alpha(f)$ crosses 0. Hence an inequality like Hypothesis 4 cannot hold with \mathcal{SRS}_0 replacing \mathcal{RS}_0 , even if $\alpha(f)$ and $\alpha(g)$ have the same sign, because the representatives in $[-1/2, 1/2)$ of $\alpha(\mathcal{RS}(f))$ and $\alpha(\mathcal{RS}(g))$ could still have opposite signs.

Let $f \in \mathcal{IS}_N \cup \mathcal{P}_N$. Then for all $n > 0$, $f_n \in \mathcal{IS}_N$. By induction for all $n \in \mathbb{N}$,

$$\Upsilon(f) = \sum_{k=0}^n \beta_{k-1}(f)C(f_k) + \beta_n(f)\Upsilon(f_{n+1}).$$

Let $n \rightarrow +\infty$. Since $\beta_n(f) \rightarrow 0$, and Υ is bounded over \mathcal{IS}_N , we get

$$\Upsilon(f) = \sum_{k=0}^{+\infty} \beta_{k-1}(f)C(f_k),$$

which is by itself an interesting formula.

Now assume either $f, g \in \mathcal{IS}_N$ or $f, g \in \mathcal{P}_N$: if $\alpha(f) = \alpha(g)$ then denoting $\beta_k = \beta_k(f) = \beta_k(g)$,

$$\Upsilon(f) - \Upsilon(g) = \sum_{k=0}^{+\infty} \beta_{k-1}(f)(C(f_k) - C(g_k)),$$

whence

$$(29) \quad |\Upsilon(f) - \Upsilon(g)| \leq K_1 \sum_{k=0}^{+\infty} \beta_{k-1} d_{\log}(f_k, g_k).$$

Otherwise, let us split the summand the usual way:

$$\begin{aligned} \beta_{k-1}(f)C(f_k) - \beta_{k-1}(g)C(g_k) &= \beta_{k-1}(f)C(f_k) - \beta_{k-1}(f)C(g_k) \\ &+ \beta_{k-1}(f)C(g_k) - \beta_{k-1}(g)C(g_k). \end{aligned}$$

Hence

$$\begin{aligned} |\beta_{k-1}(f)C(f_k) - \beta_{k-1}(g)C(g_k)| &\leq \beta_{k-1}(f) \times |C(f_k) - C(g_k)| \\ &+ |\beta_{k-1}(f) - \beta_{k-1}(g)| \times |C(g_k)|. \end{aligned}$$

Whence

$$(30) \quad |\Upsilon(f) - \Upsilon(g)| \leq \sum_{k=0}^{+\infty} \beta_{k-1}(f) |C(f_k) - C(g_k)| + K_3 \sum_{k=0}^{+\infty} |\beta_{k-1}(f) - \beta_{k-1}(g)|.$$

5.4. First term. — Here we deal with the term

$$\sum_{k=0}^{+\infty} \beta_{k-1}(f) |C(f_k) - C(g_k)|.$$

To shorten the notations we let $\alpha = \alpha(f)$, $\alpha' = \alpha(g)$, α_n and β_n associated to α , α'_n and β'_n associated to α' and

$$d'_n = d_{\log}(\alpha_n, \alpha'_n).$$

Let $n_0 \geq 0$ be the first integer so that $\alpha_{n_0}(f)$ and $\alpha_{n_0}(g)$ belong to different fundamental intervals. For some values of k we will use $|C(f_k) - C(g_k)| \leq K_1 d_{\log}(f_k, g_k)$. Recall that either $f, g \in \mathcal{IS}_N$ or $f, g \in \mathcal{P}_N$. In the second case we set $d_T((f)_0, (g)_0) = 0$. Then:

- for $k = 0$, $d_{\log}(f_k, g_k) = d_{\log}(f, g) = d_T((f)_0, (g)_0) + d'_0$,
- for $1 \leq k \leq n_0$, $d_T((f_k)_0, (g_k)_0) \leq \lambda d_T((f_{k-1})_0, (g_{k-1})_0) + K_2 d_{k-1}$

– thus by induction: for $0 \leq k \leq n_0$,

$$d_T((f_k)_0, (g_k)_0) \leq \lambda^k d_T((f)_0, (g)_0) + K_2 \sum_{m=0}^{k-1} \lambda^{k-1-m} d_m,$$

whence

$$d_{\log}(f_k, g_k) \leq \lambda^k d_T((f)_0, (g)_0) + K_2 \sum_{m=0}^{k-1} \lambda^{k-1-m} d_m + d'_k.$$

$$- d_T((\mathcal{R}\mathcal{S}f_{n_0})_0, (\mathcal{R}\mathcal{S}g_{n_0})_0) \leq \lambda d_T((f_{n_0})_0, (g_{n_0})_0) + K_2 d_{n_0}$$

Then

$$\begin{aligned} \sum_{k=0}^{n_0} \beta_{k-1} d_{\log}(f_k, g_k) &\leq \left(\sum_{k=0}^{n_0} \beta_{k-1} \lambda^k \right) d_T(f_0, g_0) + \\ &K_2 \sum_{k=0}^{n_0} \beta_{k-1} \sum_{m=0}^{k-1} \lambda^{k-1-m} d_m + \sum_{k=0}^{n_0} \beta_{k-1} d'_k. \end{aligned}$$

Let us regroup the terms according to d'_0, d'_1, d'_2, \dots and d_0, d_1, d_2, \dots :

$$\begin{aligned} \sum_{k=0}^{n_0} \beta_{k-1} d_{\log}(f_k, g_k) &\leq \left(\sum_{k=0}^{n_0} \beta_{k-1} \lambda^k \right) d_T(f_0, g_0) + \\ &\sum_{j=0}^{n_0} d'_j \beta_{j-1} + K_2 \sum_{j=0}^{n_0} d_j \left(\sum_{k=j+1}^{n_0} \beta_{k-1} \lambda^{k-1-j} \right). \end{aligned}$$

Note that since $\lambda < 1$ and $\beta_{k-1} \leq 1/2^k$, we get $\sum_{k=0}^{n_0} \beta_{k-1} \lambda^k \leq 2$. Let us bound the following factor in the preceding estimate:

$$\begin{aligned} &\sum_{k=j+1}^{n_0} \beta_{k-1} \lambda^{k-1-j} \\ &\leq \sum_{k=j+1}^{+\infty} \beta_{k-1} \lambda^{k-1-j} \\ &= \beta_{j-1} \left(\sum_{k=j+1}^{+\infty} \alpha_j \cdots \alpha_{k-1} \lambda^{k-1-j} \right) \\ &\leq \beta_{j-1} \left(\sum_{j=1}^{+\infty} 2^{-j} \right) \quad (\lambda \leq 1 \text{ and } a_j \leq 1/2) \\ &= \beta_{j-1} \end{aligned}$$

Finally we get

$$\sum_{k=0}^{n_0} \beta_{k-1} |C(f_k) - C(g_k)| \leq 2K_1 d_T(f_0, g_0) + K_1 K_2 \sum_{j=0}^{n_0} \beta_{j-1} d_j + K_1 \sum_{j=0}^{n_0} \beta_{j-1} d'_j.$$

For the remaining terms, we will make two cases:

- (A) The fundamental intervals α_{n_0} and α'_{n_0} belong to are adjacent and have symbols $(a, +)$, $(a, -)$ with the same value of a .
- (B) The other case.

In case (A) we let $n_2 = n_0 + 2$, in case (B) $n_2 = n_0 + 1$. If necessary, let us permute⁽⁸⁾ f and g , so that $\alpha_{n_0} > \alpha'_{n_0}$.

Lemma 5.2. — $|\alpha - \alpha'| \geq e^{-2C} \beta_{n_2-1}^2 / 12$.

Proof. — If the two intervals are not adjacent, then we are in case (B), so $n_2 = n_0 + 1$. Let I be the first generation interval just left to that containing α_{n_0} . Then it separates α_{n_0} from α'_{n_0} thus α and α' are separated by the fundamental interval of generation n_2 adjacent to that of α and corresponding under H_n to I . By the estimates following Lemma 2.1,

$$|\alpha - \alpha'| \geq e^{-2C} \beta_{n_2-1}^2 / 2.$$

If the intervals are adjacent but with symbols $(a, +)$, $(a + 1, -)$ we are aslo in case (B). Recall that we assumed $N \geq 3$. There is thus a pair of generation $n_0 + 2$ intervals I and I' with symbol $(2, +)$ inside $I_{n_0+1}(\alpha)$ and $I_{n_0+1}(\alpha')$ separating α from α' . The quotients $|I|/|I_{n_0+1}(\alpha)|$ and $|I'|/|I_{n_0+1}(\alpha')|$ are both bounded from below, using bounded distortion (Lemma 2.1), by e^{-C} times the value of this quotient for first generation intervals, which is equal to $1/6$. We have $|\alpha - \alpha'| \geq |I| + |I'|$ but for convenience, let us forget $|I'|$:

$$\alpha - \alpha' \geq e^{-2C} \beta_{n_2-1}^2 / 12$$

Otherwise, we are in case (A): $n_2 = n_0 + 2$. There is a fundamental interval of generation $n_0 + 2$ inside $I_{n_0+1}(\alpha)$ that separates α from α' . Hence⁽⁹⁾

$$|\alpha - \alpha'| \geq e^{-2C} \beta_{n_2-1}^2 / 2.$$

□

In both cases (A) and (B), to bound the part of the sum with $k \geq n_2$, let us use $\beta_{k-1} |C(f_k) - C(g_k)| \leq 2K_3 \beta_{k-1}$ and $\sum_{k=n_2}^{+\infty} \beta_{k-1} \leq 2\beta_{n_2-1}$, whence

$$\sum_{k=n_2}^{+\infty} \beta_{k-1} |C(f_k) - C(g_k)| \leq 4K_3 \beta_{n_2-1} \leq 4e^C K_3 \sqrt{12} |\alpha - \alpha'|^{1/2}.$$

The only term of the sum that we have not bounded is the one with $k = n_0 + 1$ in case (A). Let $\tilde{f}_{n_0+1} = \mathcal{R}\mathcal{S}f_{n_0}$ (compare with $f_{n_0+1} = \mathcal{S}\mathcal{R}\mathcal{S}f_{n_0}$) and $\tilde{g}_{n_0+1} = \mathcal{R}\mathcal{S}g_{n_0}$. Since $C(f) = C(\mathcal{S}f)$ for all f , this implies that

$$|C(f_{n_0+1}) - C(g_{n_0+1})| = |C(\tilde{f}_{n_0+1}) - C(\tilde{g}_{n_0+1})|.$$

Similarly, let $\tilde{\alpha}_{n_0+1} = 1/\alpha_{n_0} - a_{n_0+1} \in [-1/2, 1/2]$, and $\tilde{\alpha}'_{n_0+1} = 1/\alpha'_{n_0} - a'_{n_0+1} \in [-1/2, 1/2]$. This has been chosen so that: $\tilde{f}'_{n_0+1}(0) = e^{2\pi i \tilde{\alpha}_{n_0+1}}$ and $\tilde{g}'_{n_0+1}(0) =$

⁽⁸⁾This permutation is done only at this point of the article, so it is safe.

⁽⁹⁾This estimate sufficient but is far from optimal. A bound of order $\beta_{n_2-2}^2(\alpha_{n_2-1} + \alpha_{n_2-1})$ can be obtained.

$e^{2\pi i \tilde{\alpha}'_{n_0+1}}$. Let us now introduce the intermediary function $h = e^{2\pi i \tilde{\alpha}'_{n_0+1}}(\tilde{f}_{n_0+1})_0$. Then

$$|C(f_{n_0+1}) - C(g_{n_0+1})| \leq |C(\tilde{f}_{n_0+1}) - C(h)| + |C(h) - C(\tilde{g}_{n_0+1})|.$$

Then we will use the Lipschitz property of C with respect to d_{\log} . For the term $|C(h) - C(\tilde{g}_{n_0+1})|$, we obtain the upper bound $K_1(d_T((h)_0, (\tilde{g}_{n_0+1})_0) + 0)$, which equals $K_1 \times 0 + K_1 d_T((\tilde{f}_{n_0+1})_0, (\tilde{g}_{n_0+1})_0)$. For the term $|C(\tilde{f}_{n_0+1}) - C(h)|$ we obtain the upper bound $K_1 d_{\log}(\tilde{\alpha}_{n_0+1}, \tilde{\alpha}'_{n_0+1})$, which equals $K_1 d_{\log}(-\alpha_{n_0+1}, \alpha'_{n_0+1})$. Hence

$$|C(f_{n_0+1}) - C(g_{n_0+1})| \leq K_1 \left(d_T((\tilde{f}_{n_0+1})_0, (\tilde{g}_{n_0+1})_0) + d_{\log}(-\alpha_{n_0+1}, \alpha'_{n_0+1}) \right).$$

Now we can use

$$d_T((\tilde{f}_{n_0+1})_0, (\tilde{g}_{n_0+1})_0) \leq \lambda d_T((f_{n_0+1})_0, (g_{n_0+1})_0) + K_2 |\alpha_{n_0} - \alpha'_{n_0}|,$$

thus

$$d_T((\tilde{f}_{n_0+1})_0, (\tilde{g}_{n_0+1})_0) \leq \lambda^{n_0+1} d_T((f)_0, (g)_0) + K_2 \sum_{m=0}^{n_0} \lambda^{n_0-m} d_m + d'_{n_0+1}$$

which can be incorporated to the computation made earlier of the sum from $k = 0$ to n_0 . So, in case (A):

$$\begin{aligned} \sum_{k=0}^{n_0+1} \beta_{k-1} |C(f_k) - C(g_k)| &\leq 2K_1 d_T(f_0, g_0) + K_1 K_2 \sum_{j=0}^{n_0+1} \beta_{j-1} d_j \\ &+ K_1 \sum_{j=0}^{n_0+1} \beta_{j-1} d'_j + K_1 \beta_{n_0} d''_{n_0+1} \end{aligned}$$

with $d''_{n_0+1} = d_{\log}(-\alpha_{n_0+1}, \alpha'_{n_0+1})$. Let x be the touching point between the generation $n_0 + 1$ fundamental intervals containing α and α' : $|\alpha - \alpha'| = |\alpha - x| + |\alpha' - x|$. By bounded distortion (Lemma 2.1), $|\alpha - x| \geq e^{-C} \beta_{n_0}^2 \alpha_{n_0+1}$. Similarly (see the discussion following the Lemma 2.1) $|\alpha' - x| \geq e^{-2C} \beta_{n_0}^2 \alpha'_{n_0+1}$. So $|\alpha - \alpha'| \geq \beta_{n_0}^2 e^{-2C} (\alpha_{n_0+1} + \alpha'_{n_0+1})$. Using now $d_{\log}(-\alpha_{n_0+1}, \alpha'_{n_0+1}) \leq M_0 (\alpha_{n_0+1} + \alpha'_{n_0+1})^{1/2}$ we get

$$\beta_{n_0} d''_{n_0+1} \leq M_0 e^C |\alpha - \alpha'|^{1/2}.$$

Taking both cases (A) and (B) into account, we get:

$$\begin{aligned} \sum_{k=0}^{+\infty} \beta_{k-1} |C(f_k) - C(g_k)| &\leq M_1 |\alpha - \alpha'|^{1/2} + 2K_1 d_T(f_0, g_0) \\ &+ K_1 K_2 \sum_{j=0}^{+\infty} \beta_{j-1} d'_j + K_1 \sum_{j=0}^{+\infty} \beta_{j-1} d''_j \end{aligned}$$

where $M_1 = 4e^C K_3 \sqrt{12} + K_1 M_0 e^C$.

5.5. Second term. — Let us now deal with the term

$$K_3 \sum_{k=0}^{+\infty} |\beta_{k-1}(f) - \beta_{k-1}(g)|.$$

Remark. — For the modified continued fraction algorithm, each β_k is a continuous function of α , whereas it is not the case for the classical continued fractions. Note also that each β_k is in fact $1/2$ -Hölder continuous (with respect to α) with constant B_k , and that the sequence B_k is bounded but unfortunately does not tend to 0, thus $\sum B_k$ is not convergent.

The computation goes:

$$\begin{aligned} \sum_{k=0}^{+\infty} |\beta_{k-1}(f) - \beta_{k-1}(g)| &= \sum_{k=0}^{+\infty} |\beta_k(f) - \beta_k(g)| \leq \\ &\sum_{k=0}^{+\infty} \sum_{j=0}^k \alpha_0 \cdots \alpha_{j-1} |\alpha_j - \alpha'_j| \alpha'_{j+1} \cdots \alpha'_k = \\ &\sum_{j=0}^{+\infty} \alpha_0 \cdots \alpha_{j-1} |\alpha_j - \alpha'_j| \sum_{k=j}^{+\infty} \alpha'_{j+1} \cdots \alpha'_k \end{aligned}$$

With the convention that for $k = j$, $\alpha'_{j+1} \cdots \alpha'_k = 1$. Since each α'_i is $\leq 1/2$, we get

$$\sum_{k=j}^{+\infty} \alpha'_{j+1} \cdots \alpha'_k \leq 2.$$

Hence, using the following notations:

$$d_n = |\alpha_n - \alpha'_n|,$$

we obtain the following bound on the second term:

$$\begin{aligned} &K_3 \sum_{k=0}^{+\infty} |\beta_{k-1}(f) - \beta_{k-1}(g)| + K_0 |\beta_{+\infty}(f) - \beta_{+\infty}(g)| \\ &\leq 2 \max(K_3, K_0) \sum_{j=0}^{+\infty} \beta_{j-1} d_j. \end{aligned}$$

5.6. Arithmetics. — ...with a strong analytic flavor. The work presented below is not original: it is basically a simplified version of part of [MMY97], reformulated in different notations.

5.6.1. Estimates on the size of fundamental intervals. — As a trick to simplify the presentation of the next section, we introduce *extended fundamental intervals* of generation n . These are the sets on which H_{n+1} is continuous. They can also be characterized as the union of two consecutive fundamental intervals of generation n and their common boundary point, but only for those pairs whose symbols end with respectively $(a_n, +)$ and $(a_n + 1, -)$ (the rest of their symbol must be identical). Like

the fundamental intervals, their collection also form a nested sequence of partitions of the irrationals. Like above, for α irrational, we will denote

$$\tilde{I}_n(\alpha)$$

the n -th generation extended fundamental interval containing α . Another corollary of Lemma 2.1 is that for two consecutive fundamental intervals I, I' at a given level, we have:

$$e^{-2C} \leq \frac{|I'|}{|I|} \leq e^{2C}.$$

In fact we will use the following slightly better⁽¹⁰⁾ estimate, using the fact that the left derivative and the right derivative of H_n , at points where adjacent fundamental intervals meet, have equal absolute values:

$$\forall \alpha \in I, \quad e^{-2C} \leq \frac{|I'|}{\beta_{n-1}^2/2} \leq e^{2C}$$

whence

$$(31) \quad \forall \alpha \in I, \quad e^{-C} + e^{-2C} \leq \frac{|\tilde{I}_n(\alpha)|}{\beta_{n-1}^2/2} \leq e^C + e^{2C}.$$

More generally, if one has k consecutive adjacent fundamental intervals of generation n , and α belongs to one of them, then the sum Σ of their lengths satisfies

$$e^{-C} + e^{-2C} + \dots + e^{-kC} \leq \frac{\Sigma}{\beta_{n-1}^2/2} \leq e^C + e^{2C} + \dots + e^{kC}.$$

Let us now try and give various estimates of $|\alpha - \alpha'|$ in terms of their modified continued fraction expansion. Assume that $\alpha, \alpha' \in [-1/2, 1/2]$, that they are both irrationals and that $\alpha \neq \alpha'$. Let $n \geq 0$ be the first integer such that α and α' do not belong to the same generation n extended fundamental interval. If $n > 0$, from $\tilde{I}_{n-1}(\alpha) = \tilde{I}_{n-1}(\alpha')$ we get the upper bound $|\alpha - \alpha'| \leq |\tilde{I}_{n-1}(\alpha)|$ thus

$$|\alpha - \alpha'| \leq \frac{e^C + e^{2C}}{2} \beta_{n-2}^2.$$

Depending on the situations, this upper bound can be far from sharp. For a lower bound, let us consider the two cases:

– If $\tilde{I}_n(\alpha)$ and $\tilde{I}_n(\alpha')$ are not adjacent, then

$$\frac{e^{-2C} + e^{-3C}}{2} \beta_{n-1}^2 \leq |\alpha - \alpha'|.$$

– If they are adjacent, then

$$e^{-4C} (\alpha_n + \alpha'_n) \beta_{n-1}^2 \leq |\alpha - \alpha'|.$$

Those estimates can also be far from optimal.

⁽¹⁰⁾The proof could be as well carried out without this improvement

Proof. — In the first case, α and α' are separated by at least one extended interval adjacent to the one containing α .

In the second case, $[\alpha, \alpha']$ intersects between 2 and 4 fundamental intervals (on which H_n is a bijection). Let u be the common point in the boundaries of \tilde{I}_n and \tilde{I}'_n . Then $[\alpha, \alpha'] \setminus \{u\}$ splits into two components: $[\alpha, u) \subset \tilde{I}_n$, and $(u, \alpha'] \subset \tilde{I}'_n$. Consider $[\alpha, u)$. Then it is either contained in only one fundamental interval, in which case it is equal to the connected component J containing α of $H_n^{-1}((0, \alpha_n))$, and $|J| \geq e^{-C} \beta_{n-1}^{-2} \alpha_n$. Or it is not and then it contains the component J of $H^{-1}((0, 1/2))$ that is contained in \tilde{I}_n , and $|J| \geq e^{-2C} \beta_{n-1}^{-2}/2$. In both cases, $|J| \geq e^{-2C} \beta_{n-1}^{-2} \alpha_n$. Similarly, $(u, \alpha']$ contains an interval J' whose length is $\geq e^{-4C} \beta_{n-1}^{-2} \alpha'_n$. Now $|\alpha - \alpha'| \geq |J| + |J'| \geq (e^{-2C} \alpha_n + e^{-4C} \alpha'_n) \beta_{n-1}^2 \geq e^{-4C} (\alpha_n + \alpha'_n) \beta_{n-1}^2$, thus we get the lower bound $e^{-4C} (\alpha'_n + \alpha_n) \beta_{n-1}^2 \leq |\alpha - \alpha'|$. The upper bound follows from the remark after Equation 31. \square

Note that in the two cases, we get the following weaker lower bound:

$$(32) \quad 2e^{-4C} \beta_n^2 \leq |\alpha - \alpha'|.$$

5.6.2. Bounding an arithmetically defined sum. — The work of Sections 5.4 and 5.5, to prove the main theorem, we are reduced to bound the quantities $\sum_{j=0}^{+\infty} \beta_{j-1} d'_j$ and $\sum_{j=0}^{+\infty} \beta_{j-1} d_j$ by constants times $|\alpha - \alpha'|^{1/2}$. Since $d'_j > d_j$, it is enough to bound the first sum. Note that $\forall x, y \in [-1/2, 1/2]$,

$$d_{\log}(x, y) \leq M_a |x - y|^a.$$

This follows for instance from Hölder's inequality, or more simply from the following computation: if x, y of the same sign then $|\int_x^y dt/|t|| \leq \int_0^{|x-y|} dt/t$, if x, y have opposite sign then $|\int_x^y dt/|t|| \leq 2 \int_0^{|x-y|/2} dt/t$. It implies in particular, for all $a \in (0, 1)$,

$$d'_j \leq M_a |\alpha_j - \alpha'_j|^a$$

where M_a depends only on a . The first term of the sum to be bounded is $\beta_{-1} d'_0 = 1 \times d_{\log}(\alpha, \alpha')$. Applying the above with $a = 1/2$, we see that it is enough to bound the sum over $j \geq 1$. Choose your preferred $a \in (0, 1/2)$, for instance $a = 1/4$. By the discussion above, the following lemma is sufficient to conclude the proof of the main theorem.

Lemma 5.3. — *Let $a \in (1/2, 1)$. There exists $K > 0$, which depends on a , such that $\forall \alpha, \alpha' \in [-1/2, 1/2]$ that are irrational,*

$$\sum_{j=1}^{+\infty} \beta_{j-1} |\alpha_j - \alpha'_j|^a \leq K |\alpha - \alpha'|^{1/2}.$$

Remark. — Marmi, Moussa and Yoccoz realized that we cannot get a better exponent on the right hand side. They also noted that taking $a \in (0, 1/2)$ gives a function which is a -Hölder (Theorem 4.2 in [MMY97] with $\nu = 1$ and $\eta = a$).

Proof. — Assume $\alpha \neq \alpha'$ for otherwise the sum is 0. As in Section 2, let $n \geq 0$ be the first integer for which α and α' belong to different extended fundamental intervals of generation n . Let $j \geq 1$. To bound $\beta_{j-1}|\alpha_j - \alpha'_j|^a$ we will consider different cases.

Case 1, $j < n$: If $j \leq n-2$, then α and α' lie in the same j -th generation fundamental interval. By bounded distortion of $H_j : \alpha \mapsto \alpha_j$ on fundamental intervals, $|\alpha_j - \alpha'_j| \leq e^C H'_j(\alpha) |\alpha - \alpha'| = e^C \beta_{j-1}^{-2} |\alpha - \alpha'|$. For $j = n-1$, α and α' lie in the same j -th generation extended fundamental interval, whence $|\alpha_j - \alpha'_j| \leq e^{2C} \beta_{j-1}^{-2} |\alpha - \alpha'|$. In both cases, i.e. for all $j \leq n-1$, we have

$$|\alpha_j - \alpha'_j| \leq e^{2C} \beta_{j-1}^{-2} |\alpha - \alpha'|.$$

Recall that $|\alpha - \alpha'| \leq (e^C + e^{2C}) \beta_{n-2}^2 / 2$, thus

$$\begin{aligned} \frac{\beta_{j-1} |\alpha_j - \alpha'_j|^a}{|\alpha - \alpha'|^{1/2}} &\leq e^{2Ca} \left(\frac{|\alpha - \alpha'|}{\beta_{j-1}^2} \right)^{a-1/2} \leq K_4 \left(\frac{\beta_{n-2}}{\beta_{j-1}} \right)^{a-1/2} = \\ &= K_4 (\alpha_j \cdots \alpha_{n-2})^{a-1/2} \leq K_4 \left(\frac{1}{2^{(n-1)-j}} \right)^{a-1/2}. \end{aligned}$$

With $K_4 = e^{2Ca} \left(\frac{e^C + e^{2C}}{2} \right)^{a-1/2}$ and the convention that $\alpha_j \cdots \alpha_{n-2} = 1$ if $j = n-1$. Whence

$$\beta_{j-1} |\alpha_j - \alpha'_j|^a \leq \frac{K_4}{u^{n-1-j}} |\alpha - \alpha'|^{1/2}$$

with $u = 2^{a-1/2} > 1$.

Case 2, $j > n$: Then $|\alpha_j - \alpha'_j| \leq 1/2 < 1$ and $\beta_{j-1} = \alpha_0 \cdots \alpha_{j-1} = \beta_n \alpha_{n+1} \cdots \alpha_{j-1} \leq \beta_n / 2^{j-n-1} \leq \frac{e^{2C}}{\sqrt{2}} |\alpha - \alpha'|^{1/2} / 2^{j-n-1}$ by Equation (32). Whence

$$\beta_{j-1} |\alpha_j - \alpha'_j|^a \leq \frac{K_5}{2^{j-(n+1)}} |\alpha - \alpha'|^{1/2}$$

with $K_5 = e^{2C} \sqrt{2}$.

Case 3, $j = n$: We consider two sub-cases:

- \tilde{I}_n and \tilde{I}'_n are not adjacent. Then starting from $|\alpha_n - \alpha'_n| \leq 1/2 < 1$ and $|\alpha - \alpha'| \geq (e^{-2C} + e^{-3C}) \beta_{n-1}^2 / 2$, we get

$$\beta_{n-1} |\alpha_n - \alpha'_n|^a / |\alpha - \alpha'|^{1/2} \leq \sqrt{2/(e^{-2C} + e^{-3C})}$$

$$\beta_{n-1} |\alpha_n - \alpha'_n| \leq K_6 |\alpha - \alpha'|^{1/2}$$

with $K_6 = \sqrt{2/(e^{-2C} + e^{-3C})}$.

- \tilde{I}_n and \tilde{I}'_n are adjacent. Then starting from

$$|\alpha_n - \alpha'_n| \leq 1/2 < 1 \text{ and } e^{-4C} (\alpha'_n + \alpha_n) \beta_{n-1}^2 \leq |\alpha - \alpha'|$$

we get

$$\beta_{n-1} |\alpha_n - \alpha'_n|^a / |\alpha - \alpha'|^{1/2} \leq e^{2C} \frac{|\alpha_n - \alpha'_n|^a}{\sqrt{\alpha_n + \alpha'_n}}.$$

Now $|\alpha_n - \alpha'_n| \leq \max(\alpha_n, \alpha'_n)$ and $\alpha_n + \alpha'_n \geq \max(\alpha_n, \alpha'_n)$ thus

$$\beta_{n-1} |\alpha_n - \alpha'_n|^a / |\alpha - \alpha'|^{1/2} \leq e^{2C} \frac{\max(\alpha_n, \alpha'_n)^a}{\sqrt{\max(\alpha_n, \alpha'_n)}} = e^{2C} (\max(\alpha_n, \alpha'_n))^{a-1/2} \leq e^{2C}.$$

$$\beta_{n-1} |\alpha_n - \alpha'_n|^a \leq K'_6 |\alpha - \alpha'|^{1/2}$$

with $K'_6 = e^{2C}$.

By the first case:

$$\sum_{j=1}^{n-1} \beta_{j-1} |\alpha_j - \alpha'_j|^a \leq K_4 |\alpha - \alpha'|^{1/2} \sum_{s=0}^{+\infty} \frac{1}{u^s}$$

with $u = 2^{a-1/2} > 1$. By the second case:

$$\sum_{j=n+1}^{+\infty} \beta_{j-1} |\alpha_j - \alpha'_j|^a \leq K_5 |\alpha - \alpha'|^{1/2} \sum_{s=0}^{+\infty} \frac{1}{2^s}.$$

Using the three cases altogether we thus get:

$$\sum_{j=1}^{+\infty} \beta_{j-1} |\alpha_j - \alpha'_j|^a \leq K_7 |\alpha - \alpha'|^{1/2}$$

with $K_7 = \frac{K_4}{1-u} + 2K_5 + \max(K_6, K'_6)$. □

This proves the lemma, which was the last step to get the theorem.

Acknowledgement. — The first author would like to thank the Leverhulme Trust for their partial financial support while carrying out this research.

References

- [AC12] A. AVILA & D. CHERAGHI – “Statistical properties of quadratic polynomials with a neutral fixed point”, preprint, 2012.
- [Ahl63] L. V. AHLFORS – “Quasiconformal reflections”, *Acta Math.* **109** (1963), p. 291–301.
- [BC04] X. BUFF & A. CHÉRITAT – “Upper bound for the size of quadratic Siegel disks”, *Invent. Math.* **156** (2004), no. 1, p. 1–24.
- [BC06] ———, “The Brjuno function continuously estimates the size of quadratic Siegel disks”, *Ann. of Math. (2)* **164** (2006), no. 1, p. 265–312.
- [BC11] ———, “A new proof of a conjecture of Yoccoz”, *Ann. Inst. Fourier (Grenoble)* **61** (2011), no. 1, p. 319–350.
- [BC12] ———, “Quadratic julia sets with positive area”, *Ann. of Math. (2)* **176** (2012), no. 2, p. 673–746.
- [Brj71] A. D. BRJUNO – “Analytic form of differential equations. I, II”, *Trudy Moskov. Mat. Obšč.* **25** (1971), p. 119–262; *ibid.* **26** (1972), 199–239.
- [Ché08] A. CHÉRITAT – “Sur l’implosion parabolique, la taille des disques de Siegel et une conjecture de Marmi, Moussa et Yoccoz”, *Habilitation à diriger des recherches*, Université Paul Sabatier, 2008.

- [Che10] D. CHERAGHI – “Typical orbits of quadratic polynomials with a neutral fixed point: non-Brjuno type”, Preprint, 2010.
- [Che12] ———, “Typical orbits of quadratic polynomials with a neutral fixed point: Brjuno type”, Preprint, 2012.
- [IS06] H. INOU & M. SHISHIKURA – “The renormalization for parabolic fixed points and their perturbation”, Preprint, 2006.
- [Leh76] O. LEHTO – “On univalent functions with quasiconformal extensions over the boundary”, *J. Analyse Math.* **30** (1976), p. 349–354.
- [Leh87] ———, *Univalent functions and Teichmüller spaces*, Graduate Texts in Mathematics, vol. 109, Springer-Verlag, New York, 1987.
- [Mar89] S. MARMI – “Diophantine conditions and estimates for the Siegel radius: analytical and numerical results”, in *Nonlinear dynamics (Bologna, 1988)*, World Sci. Publ., Teaneck, NJ, 1989, p. 303–312.
- [Mil06] J. MILNOR – *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.
- [MMY97] S. MARMI, P. MOUSSA & J.-C. YOCOZ – “The Brjuno functions and their regularity properties”, *Comm. Math. Phys.* **186** (1997), no. 2, p. 265–293.
- [Shi98] M. SHISHIKURA – “The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets”, *Ann. of Math. (2)* **147** (1998), no. 2, p. 225–267.
- [Sie42] C. L. SIEGEL – “Iteration of analytic functions”, *Ann. of Math. (2)* **43** (1942), p. 607–612.
- [Yoc95] J.-C. YOCOZ – “Théorème de Siegel, nombres de Bruno et polynômes quadratiques”, *Astérisque* (1995), no. 231, p. 3–88, Petits diviseurs en dimension 1.

October 22, 2012

DAVOUD CHERAGHI, Mathematics Institute, University of Warwick, Coventry CV4-7AL, UK
E-mail : d.cheraghi@warwick.ac.uk

ARNAUD CHÉRITAT, CNRS/Institut de Mathématiques de Toulouse, Toulouse, France
E-mail : arnaud.cheritat@math.univ-toulouse.fr